

SUMMATION AND THE POISSON FORMULA

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Dedicated to T.N.Shorey

ABSTRACT. By giving the definition of the sum of a series indexed by a set on which a group acts, we prove that the sum of the series that defines the Riemann zeta function, the Epstein zeta function, and a few other series indexed by \mathbb{Z}^k has an intrinsic meaning as a complex number, independent of the requirements of analytic continuation. The definition of the sum requires nothing more than algebra and the concept of absolute convergence. The analytical significance of the algebraically defined sum is then explained by an argument that relies on the Poisson formula for tempered distributions.

1. INTRODUCTION

By giving the definition of a sum of a h-convergent series indexed by \mathbb{Z}^k , we will show that the sum of the series that defines the Riemann zeta function, namely

$$1 + 2^{-s} + 3^{-s} + 4^{-s} + \dots$$

has an intrinsic meaning as a complex number for every $s \neq 1$, *independent of the requirements of analytic continuation*. The definition of the sum relies only on absolute convergence and algebra. An analytic meaning for this algebraically defined sum is supplied by the Poisson formula. In our context, it is better to rewrite this formula in the form below:

$$\mathcal{L}f := -\hat{f}(0) + \sum_{0 \neq n \in \mathbb{Z}^k} f(n) = -f(0) + \sum_{0 \neq n \in \mathbb{Z}^k} \hat{f}(n) = \mathcal{L}\hat{f}.$$

It is shown that the linear functional \mathcal{L} , defined originally on the Schwartz space of functions on \mathbb{R}^k with rapid decay, extends in a natural manner to a linear functional on the subspace $\mathcal{H}(\mathbb{R}^k)$ of the space of tempered distributions defined in 6.1. These distributions are C^∞ on the region $0 \neq x \in \mathbb{R}^k$. The value of $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$ assigned by algebraic considerations for *homogeneous* $f \in \mathcal{H}(\mathbb{R}^k)$ is seen to coincide with $\mathcal{L}f$.

We begin with the definition of t-summability of a series indexed by an Abelian-group Γ , and then proceed to h-summability.

Given a function $a : \Gamma \rightarrow \mathbb{C}$, we want to define $\sum_{\gamma \in \Gamma} a(\gamma) \in \mathbb{C}$ in a translation-invariant manner. Equivalently, we want the identity

$$(1) \quad \left(\sum_{\gamma \in \Gamma} c(\gamma) \right) \left(\sum_{\gamma \in \Gamma} a(\gamma) \right) = \sum_{\gamma \in \Gamma} c * a(\gamma)$$

for every finitely supported $c : \Gamma \rightarrow \mathbb{C}$, where $c * a$ denotes the convolution of c with a . Now the above formula is valid when the series $\sum_{\gamma \in \Gamma} a(\gamma)$ is absolutely convergent. We

are thus led to the following definition: the series $\sum_{\gamma \in \Gamma} a(\gamma)$ is translation-summable, or simply t-summable (or t-convergent), if there is a finitely supported $c : \Gamma \rightarrow \mathbb{C}$ so that $0 \neq \sum_{\gamma \in \Gamma} c(\gamma)$ and the series $\sum_{\gamma \in \Gamma} c * a(\gamma)$ is absolutely convergent. The sum of the series is then defined by the formula below:

$$(2) \quad \sum_{\gamma \in \Gamma} a(\gamma) = \left(\sum_{\gamma \in \Gamma} c(\gamma) \right)^{-1} \sum_{\gamma \in \Gamma} c * a(\gamma).$$

The commutativity of Γ ensures the properties below:

- (a) the sum of the above t-summable series given in (2) is independent of the choice of c .
- (b) the collection of $a : \Gamma \rightarrow \mathbb{C}$ for which $\sum_{\gamma \in \Gamma} a(\gamma)$ is t-summable is a linear subspace that is stable under translation by all $\gamma \in \Gamma$,
- (c) $a \mapsto \sum_{\gamma \in \Gamma} a(\gamma)$ is a Γ -invariant linear functional defined on the space of all t-summable $a : \Gamma \rightarrow \mathbb{C}$. In particular, if $\sum_{\gamma \in \Gamma} a(\gamma)$ is t-convergent, then so is $\sum_{\gamma \in \Gamma} c * a(\gamma)$ for every finitely supported function $c : \Gamma \rightarrow \mathbb{C}$, and equation (1) holds.

It is shown in section 3 that the sum of a t-convergent series behaves well with respect to parameters. The main example of a t-summable series for $\Gamma = \mathbb{Z}^k$ that we are concerned with follows

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

For $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$ and $z = (z_1, \dots, z_k) \in \mathbb{T}^k$, we put $z^n = z_1^{n_1} z_2^{n_2} \dots z_k^{n_k}$.

Given a subset $M \subset \mathbb{Z}^k$ and $a : M \rightarrow \mathbb{C}$, we define $a' : \mathbb{Z}^k \rightarrow \mathbb{C}$ by

$$a'(n) = a(n) \text{ if } n \in M \text{ and } a'(n) = 0 \text{ if } n \notin M.$$

When $\sum_{n \in \mathbb{Z}^k} a'(n)$ is t-summable, we will simply say that $\sum_{n \in M} a(n)$ is t-summable.

A function $f : \{0 \neq x \in \mathbb{R}^k\} \rightarrow \mathbb{C}$ is homogeneous of type $(s, \epsilon) \in \mathbb{C} \times \{\pm 1\}$ if

$$(3) \quad f(tx) = t^s f(x) \text{ and } f(-x) = \epsilon f(x) \text{ for all } t > 0, 0 \neq x \in \mathbb{R}^k$$

Theorem 1.1. *Let $f : \{0 \neq x \in \mathbb{R}^k\} \rightarrow \mathbb{C}$ be a C^∞ function homogeneous of type $(-s, \epsilon)$. Then the series*

$$(4) \quad F_f(x, z) = \sum_{0 \neq n \in \mathbb{Z}^k} f(x + n) z^n \text{ is t-summable } \forall 1 \neq z \in \mathbb{T}^k, \forall x \in D$$

where $D = \{x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : |x_i| < 1 \forall 1 \leq i \leq k\}$. Furthermore the sum of this series gives a C^∞ function on the domain $D \times (\mathbb{T}^k \setminus \{1\})$.

This result is elementary: it relies only on the fundamental theorem of calculus and the finiteness of the integral $\int_{\mathbb{R}^k} (1 + \|x\|)^{-k-1}$.

Take $k = 1$ in the theorem. Let $f(x) = |x|^{-s}$ if $\epsilon = 1$. If $\epsilon = -1$, we take $f(x) = x|x|^{-s-1}$. Taking suitable linear combinations of the $F_f(0, z)$ where $z^m = 1, z \neq 1$, we obtain the value of $L(\chi, s)$ for every *nontrivial* Dirichlet character and every $s \in \mathbb{C}$. The holomorphicity of $s \mapsto L(\chi, s)$ is a relatively simple matter. The vanishing of $L(\chi, s)$ at integers $s \leq 0$ such that $\chi(-1) = (-1)^s$ follows in an equally simple manner (see 3.4 and 4.5). The familiar series obtained from $k = 1, z = -1$

$$1 - 2^{-s} + 3^{-s} - 4^{-s} \dots$$

was shown by J.Sondow (see [12]) to be an entire function of s by the finite difference method in a manner essentially identical to the one given here.

All this does not cover the Riemann zeta function; here we require the sum of the series for $z = 1$. With f, F_f as in thm. 1.1, it would be natural to obtain the ‘value’ of $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$ by evaluating the limit $\lim_{z \rightarrow 1} F_f(0, z)$, but this limit does not exist in general. However this problem can be resolved algebraically in the following manner.

Let $m > 1$ be a natural number. Let $\mathbb{T}^k(m) = \{z \in \mathbb{T}^k : z^m = 1\}$. The formal identity with f, F_f as is in (4)

$$\sum_{\lambda \in \mathbb{T}^k(m)} F_f(0, \lambda \cdot z) = m^{k-s} F_f(0, z^m) \forall z \in \mathbb{T}^k \text{ with } z^m \neq 1$$

holds for t-convergent sums. We rewrite the above as

$$F_f(0, z) - m^{k-s} F_f(0, z^m) = - \sum_{1 \neq \lambda \in \mathbb{T}^k(m)} F_f(0, \lambda \cdot z).$$

The term on the right is defined for all z in a neighborhood of $1 \in \mathbb{T}^k$. This suggests that the value of $F_f(0, 1)$ can be ‘forced’ by setting $z = 1$ in the above formula when $s \neq k$ by choosing $m \in \mathbb{N}$ so that $m^{s-k} \neq 1$:

$$(5) \quad \sum_{0 \neq n \in \mathbb{Z}^k} f(n) = -(1 - m^{k-s})^{-1} \sum_{1 \neq \lambda \in \mathbb{T}^k(m)} F_f(0, \lambda \cdot)$$

The procedure above leads to the definition of h-summability (or h-convergence) of a series. It is in fact an iteration of the method that defines t-summability. The precise definition is given in the next section. The series $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$ is h-summable when $(s, \epsilon) \neq (k, 1)$ and its sum is given by (5) when $s \neq k$. When $\epsilon = -1$, the sum of the series is zero.

Both t-summability and h-summability can be used to deduce some standard results on analytic continuation (see 3.7, 3.10 and 4.6) and some others as well. For instance, the analytic continuation of $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)^{-s}$ gives rise to the Epstein zeta function when f is a positive definite quadratic form. When $f(x) = x_1^4 + x_2^4 + \dots + x_k^4$, the analytic continuation of the above sum is perhaps¹ ancient and forgotten. Both these examples are dealt with in prop. 4.6 by the same method. Theorem 4.7 deals with the same sum in the inhomogeneous case, but part (2) of that result has a surprise. Dinesh Thakur suggested to the author a comparison with the methods listed in G.H.Hardy’s book [6]. This remains to be done.

Theorems on analytic continuation abound in the theory of automorphic forms, while 3.10 and 4.6 only covers the case of the Eisenstein series for $\mathrm{GL}_2(\mathbb{Q})$ and the very first cases of Eisenstein series for $\mathrm{GL}_k(\mathbb{Q})$ for $k > 2$. A point of difference is that such results for automorphic forms are stated only for K -finite vectors, and not for C^∞ vectors, as we have. It is unclear to what extent intrinsic summability definitions can be pushed to ‘explain’ results on analytic continuation.

The main result of this paper (thm. 1.2 and cor. 1.3 below) is the relation between the sum in (5) and the behaviour of $F_f(x, z)$ in a neighbourhood of $(0, 1) \in D \times \mathbb{T}^k$, with notation as in theorem 1.1. We set up the requisite notation for Fourier transforms and the Poisson formula, needed for this purpose.

¹Assistance in attributing results correctly will be appreciated and recorded.

$$(6) \quad \psi(x, y) = \exp 2\pi\sqrt{-1}(x_1y_1 + x_2y_2 + \dots + x_ky_k)$$

for $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in \mathbb{R}^k$. We have the Fourier transform

$$(7) \quad \hat{h}(y) = \int_{\mathbb{R}^k} h(x)\psi(x, y)dx_1dx_2\dots dx_k \text{ for } h \in L^1(\mathbb{R}^k)$$

We will write

$$(8) \quad \mathbf{e}_k : \mathbb{R}^k \rightarrow \mathbb{T}^k \text{ for } (x_1, \dots, x_k) \mapsto (\exp 2\pi\sqrt{-1}x_1, \dots, \exp 2\pi\sqrt{-1}x_k)$$

It is well known that a $C^\infty f$ homogeneous of type $(-s, \epsilon)$ extends uniquely as a homogeneous tempered distribution (see [5], vol.1, chapter 7) as long as

$$(9) \quad (-s, \epsilon) \notin \{(e + k, (-1)^e) : e = 0, 1, 2, \dots\}.$$

Its Fourier transform \hat{f} is then C^∞ on $\{0 \neq y \in \mathbb{R}^k\}$ and is homogeneous of type $-(k - s, \epsilon)$. The notation of (6) and (8) is used in the theorem below.

Theorem 1.2. *With f, F_f, D be as in thm. 1.1, assume furthermore that both $(-s, \epsilon)$ and $-(k - s, \epsilon)$ satisfy (9). Then we have:*

- (1) $F_f(x, \mathbf{e}_ky) - \psi(-x, y)\hat{f}(y) = -f(x) + \psi(-x, y)F_{\hat{f}}(y, -\mathbf{e}_kx)$ if $0 \neq x, y \in D$
- (2) *There is a C^∞ function $F_f^{reg} : D \times D \rightarrow \mathbb{C}$ that restricts to either of the functions in part (1) above.*
- (3) $F_f^{reg}(0, 0)$ equals $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$ as defined by (5)
- (4) $\sum_{0 \neq n \in \mathbb{Z}^k} f(n) = \sum_{0 \neq n \in \mathbb{Z}^k} \hat{f}(n)$

Note that part (1) of the theorem is simply the Poisson formula for the function h given by $h(w) = f(w + x)\psi(w, y)$. This requires a proof because at least one of the two series there is not absolutely convergent. The function on the left in (1) has domain $D \times (D \setminus \{0\})$, while the one on the right has $(D \setminus \{0\}) \times D$ as domain. The equality therefore thus extends the domain of this function to their union, which is the complement of $(0, 0)$ in $D \times D$. This falls short of part (2), which is the essence of the theorem.

Part (2) with $x = 0$ and part (3) combine to give:

Corollary 1.3. $y \mapsto F_f(0, \mathbf{e}_ky) - \hat{f}(y)$ defined for $0 \neq y \in D$ extends to a C^∞ function on D whose value at zero is given by $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$.

We think of the above corollary as the analytic significance of the algebraically defined sum.

Thm. 1.2(4) follows from interchanging the roles of f and \hat{f} . When $k = 1$, this is the functional equation of the Riemann zeta function. The environment of thm. 1.2 for $k = 1$ has been subjected to extensive study. Our functions $F_f(x, \exp 2\pi\sqrt{-1}y)$ are closely related to the Lerch zeta function

$$\phi(y, x, s) = \sum_{n=0}^{\infty} (x + n)^{-s} \exp 2\pi\sqrt{-1}ny$$

whose properties have been investigated in recent times by J.Lagarias and W.Li. The reader can gather its history from their paper [8]. The functional equation of the

Lerch zeta function (see (1.4), page 161, [2]) is essentially equivalent to thm. 1.2(1) for $k = 1$. It was proved by Lerch in 1887. Another proof of this theorem is given by Apostol [2] where he follows Riemann's method. As is well known, this method begins by expressing \int_0^∞ as $\int_0^1 + \int_1^\infty$, then converts \int_0^1 to \int_1^∞ by a change of variables, and proceeds. We do not follow this method, and rely instead on conventional methods in distributions, namely the use of cut-off functions. A proof of the functional equation of the zeta function that also relies on distributions, similar but not identical to ours, is due to S. Miller and W. Schmid in a paper [11] that has far wider goals and applications. The paper [4] of P. Gerardin and W. Li interprets the functional equation of the zeta function as an equality of tempered distributions.

The plan of the paper is as follows. The next section has the definitions of t -summability and h -summability. Sections 3 and 4 have some examples of t - and h -summable series. Section 5 is essentially a review of the Poisson formula for distributions based on ideas borrowed from [16], [9] and [10], followed by some simple analysis of the singularities. This discussion is applied to a class of distributions $\mathcal{H}(\mathbb{R}^k)$ defined in section 6. The main result there is Theorem 6.6. The proofs of the theorems as stated in the introduction fall out easily from this result; they are summed up in 6.9. This section closes with a definition of t -integrability of distributions on \mathbb{R}^k .

The last section discusses the more general definition of the sum of a series indexed by a set X which a group G acts. This definition is more stringent than h -summability, but has the advantage of getting rid of the unpleasant behaviour of 4.7(2). On the other hand, if we were to adopt the H -summability of 7.5 we would have to regard the Riemann zeta function as a function with a simple pole at the negative odd integers with residue zero!

A word of apology regarding notation. The Euclidean space figuring in sections 3 and 4 is invariably \mathbb{R}^k . Sections 5 and 6 have finite dimensional real vector spaces X and X' , and the ' k 's of those sections are variable non-negative integers. Finally, in sections 2 and 7, X denotes a set equipped with the action of a group G .

2. DEFINITION OF t -SUMMABILITY AND h -SUMMABILITY

Set-up 2.1. Given

- (a) a commutative ring A , a field k , and a ring homomorphism $\epsilon : A \rightarrow k$ where k is a field,
- (b) a left A -module homomorphism $I : M \rightarrow k$, and
- (c) an inclusion $M \hookrightarrow N$ of left A -modules,

we define the *canonical extension* $I_c : M_c \rightarrow k$ below.

Let $S = \{a \in A : \epsilon(a) \neq 0\}$. This is a multiplicative subset of A . We obtain $S^{-1}\epsilon : S^{-1}M \rightarrow k$. Let $i : N \rightarrow S^{-1}N$ denote the homomorphism $n \mapsto \frac{n}{1}$ for all $n \in N$. Note that $S^{-1}M$ is a subset of $S^{-1}N$. We define M_c by $M_c = \{n \in N : i(n) \in S^{-1}M\} = \{n \in N : \exists s \in S \text{ so that } sn \in M\}$.

Now $i : N \rightarrow S^{-1}N$ restricts to $j : M_c \rightarrow S^{-1}M$. We define I_c to be the composite

$$M_c \xrightarrow{j} S^{-1}M \xrightarrow{S^{-1}\epsilon} k$$

This will be applied in the following manner. For an abstract group G , its group-algebra is denoted by $\mathbb{C}[G]$ and $\epsilon : \mathbb{C}[G] \rightarrow \mathbb{C}$ denotes the augmentation. With

$B = \mathbb{C}[G]$, we are given the data (b) and (c) as above: an inclusion $M \hookrightarrow N$ of $\mathbb{C}[G]$ -modules, and a G -invariant linear functional $I : M \rightarrow \mathbb{C}$. We do not assume that G is commutative, but that there are commutative subgroups $H_1, H_2, \dots, H_n \subset G$ with the property that H_j is contained in the normaliser of H_i whenever $1 \leq i < j \leq n$. It follows that $G_k = H_k H_{k+1} \dots H_n$ is a subgroup of $N(H_k)$, the normaliser of H_k in G , for all $k = 1, 2, \dots, n$.

The commutative ring $A = \mathbb{C}[H_1]$ and the data $(M \hookrightarrow N, I : M \rightarrow \mathbb{C})$ of A -module homomorphisms produce the canonical extension, denoted by $I_1 : M_1 \rightarrow \mathbb{C}$. Let $N(H_1)$ denote the normaliser of H_1 in G . It is clear from the construction that M_1 is a $N(H_1)$ -module and that I_1 is a $N(H_1)$ -invariant linear functional. Because H_2 is contained in $N(H_1)$, we may repeat the procedure with $A = \mathbb{C}[H_2]$ and the data $(M_1 \hookrightarrow N, I_1 : M_1 \rightarrow \mathbb{C})$ of A -module homomorphisms, and denote the resulting canonical extension by (M_2, I_2) . Proceeding inductively we obtain

- (1) the chain $M \subset M_1 \subset M_2 \subset \dots \subset M_n \subset N$ where M_k is a G_k -module for all $k = 1, 2, \dots, n$, and
- (2) G_k -invariant linear functionals $I_k : M_k \rightarrow \mathbb{C}$ for $k = 1, 2, \dots, n$ so that $I_1|_M = I$, and $I_k|_{M_{k-1}} = I_{k-1}$ if $k = 2, 3, \dots, n$. Taking $k = n$, we see that M_n is a H_n -module and $I_n : M_n \rightarrow \mathbb{C}$ is H_n -invariant.

We apply the above construction in the following manner. Let X be a set equipped with G -action. We take

$$N = \mathbb{C}^X, M = L^1(X), I = \Sigma_X \text{ given by } Ia = \Sigma_X a = \sum_{x \in X} a(x) \forall a \in L^1(X)$$

For **t-summability** we take $X = \Gamma$ a commutative group, $G = \Gamma$ acting on itself by translation, $n = 1$, and $H_1 = G$. The canonical extension (M_1, I_1) by abuse of notation, is denoted by $(L_t^1(\Gamma), \Sigma_\Gamma)$. As already remarked, Σ_Γ is now a Γ -invariant linear functional on $L_t^1(\Gamma)$. This definition of t-summability is exactly the same as that of the introduction.

The $(M \hookrightarrow N, I : M \rightarrow \mathbb{C})$ remain unchanged for **h-summability**, but now we assume that Γ is a R -module, where R is a commutative ring. The group of units R^\times of R acts on Γ . Let G be the semidirect product of Γ and R^\times . We take $n = 2$, $H_1 = \Gamma$ and $H_2 = R^\times$. The canonical extension (M_2, I_2) is denoted by $(L_h^1(\Gamma), \Sigma_\Gamma)$. The above discussion is summarised in the proposition below.

Proposition 2.2. *Let Γ be a R -module, where R is a commutative ring.*

The spaces $L^1(\Gamma) \subset L_t^1(\Gamma) \subset L_h^1(\Gamma) \subset \mathbb{C}^\Gamma$ are all stable under the natural action of R^\times .

Furthermore $\Sigma_\Gamma : L_h^1(\Gamma) \rightarrow \mathbb{C}$ is a R^\times -invariant linear functional.

The space $L_t^1(\Gamma)$ is stable under translation by Γ and $\Sigma_\Gamma|_{L_t^1(\Gamma)}$ is a Γ -invariant linear functional.

3. t-SUMMABILITY

A commutative discrete group Γ remains fixed throughout this section. The ring structure of the group algebra $\mathbb{C}[\Gamma]$ is given by convolution. For $d \in \mathbb{C}[\Gamma]$, we will denote the n -th power of d in this ring, namely $d * d * \dots * d$ “ n times” by $d^{(n)}$.

$\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$. $\Gamma^* = \text{Hom}(\Gamma, \mathbb{T})$ is the Pontrjagin dual of a commutative

group Γ . Given functions $a, b : \Gamma \rightarrow \mathbb{C}$, pointwise multiplication gives $a.b : \Gamma \rightarrow \mathbb{C}$ and convolution (when defined) gives another function $a * b : \Gamma \rightarrow \mathbb{C}$. We shall simply say a is t-summable when the series $\sum_{\gamma \in \Gamma} a(\gamma)$ has that property, and then denote the sum of the series simply by Σa . The lemma below demonstrates that t-summability behaves well.

Lemma 3.1. *Let $a : \Gamma \rightarrow \mathbb{C}$ be a function and let $\chi_0 \in \Gamma^*$. Assume that the series $\Sigma(a.\chi_0)$ is t-summable. Then the series $\Sigma(a.\chi)$ is t-summable for all χ in some neighborhood U of χ_0 in Γ^* . Furthermore, $\chi \mapsto \Sigma(a.\chi)$ gives a continuous function on U .*

Proof. The t-summability of $a.\chi_0$ says there is some $c \in \mathbb{C}[\Gamma]$, thus c is a function on Γ with finite support, so that

$$(i) \Sigma c \neq 0 \text{ and } (ii) b = c * (a.\chi_0) \in L^1(\Gamma).$$

Let $h(\chi) = \Sigma(c.\chi)$ and $g(\chi) = \Sigma(b.\chi)$ for all $\chi \in \Gamma^*$. Both g and h are continuous functions on Γ^* . Let V be the open subset of Γ^* given by $h(\chi) \neq 0$. By assumption (i) above, $1 \in V$. Now convolution commutes with pointwise multiplication by a character. From this we see that $a.\chi_0.\chi$ is t-summable when $\chi \in V$ and also that $\Sigma(a.\chi_0.\chi) = g(\chi)/h(\chi)$ for all $\chi \in V$. □

Recall that the n -th power of d in the group-algebra $\mathbb{C}[\Gamma]$ is denoted by $d^{(n)}$.

Lemma 3.2. *Let $c = \sum_{i=1}^k (1 - \gamma_i) * (1 - \gamma_i^{-1}) \in \mathbb{C}[\Gamma]$. Assume that $\gamma_1, \gamma_2, \dots, \gamma_k$ are generators of Γ . Let $a : \Gamma \rightarrow \mathbb{C}$ be a function. For the statements below, (1) \implies (2) \implies (3)*

- (1) *there is a natural number n so that $(1 - \gamma_i)^{(n)} * a \in L^1(\Gamma)$ for all $i = 1, \dots, k$.*
- (2) *there is a natural number n so that $c^{(n)} * a \in L^1(\Gamma)$*
- (3) *$a.\chi$ is t-summable for all $1 \neq \chi \in \Gamma^*$.*

Proof. We see that $c^{(n)}$ is in the ideal generated by $\{(1 - \gamma_i)^{(2n)} : i = 1, 2, \dots, k\}$ and this proves the first implication. We note next that $h(\chi) = \Sigma(c.\chi)$ is the sum of $|1 - \chi(\gamma_i)|^2$ taken over $i = 1, 2, \dots, k$, and thus $h(\chi) > 0$ if $\chi \neq 1$. It follows that $\Sigma c^{(n)}.\chi = h(\chi)^n > 0$ for such χ . Because $(c^{(n)} * a).\chi = (c^{(n)}.\chi) * (a.\chi)$, we conclude the t-summability of $a.\chi$ when $\chi \neq 1$. □

Lemmas 3.3 and 3.4 are unimportant for the moment, for they will be appealed to much later.

Lemma 3.3. *Assume that $a : \Gamma \rightarrow \mathbb{C}$ satisfies condition (1) of lemma 3.2. Let $N \in \mathbb{N}$ and let $p : \Gamma \rightarrow \Gamma/N\Gamma$ denote the projection. Let $b : \Gamma/N\Gamma \rightarrow \mathbb{C}$ be a function satisfying $\Sigma_{\Gamma/N\Gamma} b = 0$.*

(1) *Then the series $\Sigma(b \circ p).a$ is t-summable.*

Let $w_i = 1 + \gamma_i + \dots + \gamma_i^{(N-1)}$ and let $w = w_1^{(n)} * \dots * w_k^{(n)}$.
 (2) *Then $\Sigma w = N^{nk} > 0$ and $w * ((b \circ p).a)$ is in $L^1(\Gamma)$.*

Proof. The hypothesis $\Sigma_{\Gamma/N\Gamma} b = 0$ implies that b is a linear combination of nontrivial characters χ of $\Gamma/N\Gamma$. Thus (1) follows from the previous lemma. It suffices to prove

(2) for $b = \chi$ with χ as above. Now (1) of lemma 3.2 implies that $(1 - \chi(p\gamma_i)\gamma_i)^{(n)} * (a.(\chi \circ p))$ is in $L^1(\Gamma)$. Choose i so that $\chi(p\gamma_i) \neq 1$. Then $1 - \chi(p\gamma_i)\gamma_i$ divides w_i in the group algebra. It follows that $w * (a.(\chi \circ p))$ is in $L^1(\Gamma)$. This proves (2). \square

Lemma 3.4. *With the assumptions of lemma 3.2, assume there is some natural number n so that $(1 - \gamma_i)^{(n)} * a = 0$ for all $i = 1, 2, \dots, k$. Then $a.\chi$ is t -summable for all $1 \neq \chi \in \text{Hom}(\Gamma, \mathbb{C}^\times)$, and in fact $\Sigma a.\chi = 0$.*

Proof. This follows from $0 = ((1 - \gamma_i)^{(n)} * a).\chi = (1 - \chi(\gamma_i)\gamma_i)^{(n)} * (a.\chi)$ and $\Sigma(1 - \chi(\gamma_i)\gamma_i)^{(n)} = (1 - \chi(\gamma_i))^n$, noting that $1 \neq \chi(\gamma_i)$ for some i , once it assumed that $\chi \neq 1$. \square

Remark 3.5. It will be necessary to state everything with parameters, so we set up notation and recall some simple facts from a first course in Analysis such as [1],[13]. $\|a\|_1$ stands for $\Sigma|a|$, for all $a : \Gamma \rightarrow \mathbb{C}$.

Given a function $a : W \times \Gamma \rightarrow \mathbb{C}$, we put

$$a_w(\gamma) = a(w, \gamma) \quad \forall w \in W, \gamma \in \Gamma, \text{ and define } \|a\|' = \sup\{\|a_w\|_1 : w \in W\}.$$

$$\text{If } \|a\|' < \infty \text{ let } G_a(w, \chi) = \Sigma a_w.\chi = \sum_{\gamma \in \Gamma} a(w, \gamma)\chi(\gamma), \text{ then } |G_a(w, \chi)| \leq \|a\|'.$$

In the three statements below, it is assumed that $\|a\|' < \infty$.

- (1) If a is continuous, then $G_a : W \times \Gamma^* \rightarrow \mathbb{C}$ is also continuous.
- (2) If $W \subset \mathbb{C}$ is open and $w \mapsto a(w, \gamma)$ is holomorphic for every $\gamma \in \Gamma$, then $w \mapsto G_a(w, \chi)$ is holomorphic for every $\chi \in \Gamma^*$.
- (3) If $W \subset \mathbb{R}^m$ is open, if $v \in \mathbb{R}^m$, and if the directional derivative $\partial_v a$ of $w \mapsto a(w, \gamma)$ exists, is continuous, and $\|\partial_v a\|' < \infty$, then the directional derivative $\partial_v G_a$ of the function $w \mapsto G_a(w, \chi)$ exists for all $w \in W, \chi \in \Gamma^*$ and is in fact given by $\partial_v G_a = G_b$ where $b = \partial_v a$.

For the rest of this section, we take $\Gamma = \mathbb{Z}^k$ and $\Gamma^* = \mathbb{T}^k$. The given basis of \mathbb{Z}^k , namely $(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$, will be denoted e_1, e_2, \dots, e_k . Given $a : \mathbb{Z}^k \rightarrow \mathbb{C}$ we put $\Delta_i a = (1 - e_i) * a$. Thus $\Delta_i a(n) = a(n) - a(n - e_i)$ for all $n \in \mathbb{Z}^k$. Iterating this operator m times we have $(1 - e_i)^{(m)} * a = \Delta_i^m a$. We wish to find sufficient conditions that ensure $\Delta_i^m a \in L^1(\mathbb{Z}^k)$ for all $1 \leq i \leq k$ in order to appeal to lemma 3.2. We do so when $a = f|_{\mathbb{Z}^k}$ and $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is a C^∞ function. We use the same notation Δ_i for functions on \mathbb{R}^k as well. Expressing $\Delta_i f$ as the integral of $\partial_i f$ and iterating this procedure, we get

$$\Delta_i^m f(x) = \int_0^1 \dots \int_0^1 \partial_i^m f(x - (t_1 + t_2 + \dots + t_m)e_i) dt_1 dt_2 \dots dt_m$$

which gives

$$(10) \quad |\Delta_i^m f(x)| \leq \sup\{|\partial_i^m f(x - te_i)| : 0 \leq t \leq m\}.$$

For the rest of this section, for $g : \mathbb{R}^k \rightarrow \mathbb{C}$ we set

$$(11) \quad \|g\|'' = \sup\{(1 + \|x\|)^{k+1} |g(x)| : x \in \mathbb{R}^k\}$$

$\|x\|^2 = \langle x, x \rangle$ is the standard Euclidean norm on \mathbb{R}^k . We obtain a constant $C(m, k)$

$$(12) \quad \sum_{n \in \mathbb{Z}^k} \sup\{|g(n + y)| : y \in \mathbb{R}^k, \|y\| \leq m\} \leq C(m, k) \|g\|''$$

Putting all the above together we see

$$(13) \quad \sum_{n \in \mathbb{Z}^k} |\Delta_i^m f(n)| \leq C(m, k) \|\partial_i^m f\|''$$

so all we require is the finiteness of $\|\partial_i^m f\|''$ for all $1 \leq i \leq k$ and for some m with $\|\cdot\|''$ as given in (11). A convenient class of functions which satisfies this condition is given below.

Definition 3.6. A \mathbb{C} -valued C^∞ function f defined on the complement of a compact $K \subset \mathbb{R}^k$ is \mathcal{H}_∞ if there is some $p \in \mathbb{R}$ for which

$$(14) \quad \|x\|^r \partial_v^r f(x) \text{ is } O(\|x\|^p) \text{ as } \|x\| \rightarrow \infty, \forall v \in \mathbb{R}^k, \forall r \geq 0$$

where ∂_v denotes the directional derivative.

Let $f : W \times (\mathbb{R}^k \setminus K) \rightarrow \mathbb{C}$ be a function so that f_w is C^∞ for all $w \in W$, (where $f_w(x) = f(w, x)$ for all $w \in W, x \in \mathbb{R}^k$). We say f is uniformly \mathcal{H}_∞ if there

$$(15) \quad \sup\{\|x\|^{r-p} |\partial_v^r f(w, x)| : w \in W, x \in \mathbb{R}^k, x \notin K'\} < \infty, \forall v \in \mathbb{R}^k, \forall r \geq 0$$

for some $p \in \mathbb{R}$ and some compact subset K' of \mathbb{R}^k that contains K .

Note that if $f : \mathbb{R}^k \rightarrow \mathbb{C}$ is C^∞ , (14) is equivalent to (16) below

$$(16) \quad \sup\{(1 + \|x\|)^{r-p} |\partial_v^r f(x)| : x \in \mathbb{R}^k\} < \infty \forall v \in \mathbb{R}^k, \forall r \geq 0$$

We are ready to state and prove the main result of this section. The notation Σ' below stands for the sum taken over $\{n \in \mathbb{Z}^k, n \notin K\}$. Precisely, the n -th term of the series is defined to be zero when $n \in K \cap \mathbb{Z}^k$, and the resulting series whose terms are indexed by \mathbb{Z}^k is tested for t -convergence.

Theorem 3.7. (A) Assume $f : \mathbb{R}^n \setminus K \rightarrow \mathbb{C}$ is \mathcal{H}_∞ . Then the series $\sum_{n \in \mathbb{Z}^n}' f(n) z^n$ is t -convergent for all $1 \neq z \in \mathbb{T}^k$. The sum of the series is continuous on the domain $1 \neq z \in \mathbb{T}^k$.

(B) Assume that $f : W \times (\mathbb{R}^k \setminus K) \rightarrow \mathbb{C}$ is uniformly \mathcal{H}_∞ . Denote the sum of the t -convergent series $\sum_{n \in \mathbb{Z}^k} f(w, n) z^n$ by $G_f(w, z)$ for all $w \in W, 1 \neq z \in \mathbb{T}^k$. If f is continuous, so is G_f .

(C) If in addition, $W \subset \mathbb{C}$ is open and $w \mapsto f(w, x)$ is holomorphic for every $x \in \mathbb{R}^k, x \notin K$, then $w \mapsto G_f(x, w)$ is a holomorphic function of $w \in W$ for every $1 \neq z \in \mathbb{T}^k$.

Proof. We first prove all three parts of the theorem under the assumption that K is empty.

In (A) we therefore assume that f is defined on all of \mathbb{R}^k . The estimate (16) is valid now. With p as in that estimate, we choose any m so that $m - p \geq (k + 1)$. It follows that $\|\partial_i^m f\|'' < \infty$ for all $1 \leq i \leq k$. The bound (13) then shows that $\Delta_i^m f|_{\mathbb{Z}^k} \in L^1(\mathbb{Z}^k)$ for all $1 \leq i \leq k$. The “(1) implies (3)” of lemma 3.2 proves the t -convergence of the given series. By lemma 3.1 we see that the sum of this series is continuous.

For parts (B) and (C), we once again choose m in exactly the same manner, but with p as in (15). We deduce that $\{\|\Delta_i^m f_w|_{\mathbb{Z}^k}\|_1 : w \in W, 1 \leq i \leq k\}$ is bounded

above. By remark 3.5, we see that u_i defined by

$$u_i(w, z) = \sum_{n \in \mathbb{Z}^k} \Delta_i^m f(w, n) z^n \text{ for all } w \in W, z \in \mathbb{T}^k$$

is a continuous function (and holomorphic in w under the assumptions of (C)). Because $(1 - z_i)^m G_f(w, z) = u_i(w, z)$ for all $1 \neq z \in \mathbb{T}^k$ and for all $1 \leq i \leq k$, we see that the function $G_f(w, z)$ has the same property. This completes the proof of the theorem when K is empty.

We now come to the general case. Choose a test function $\phi \in C_c^\infty(\mathbb{R}^k)$ satisfying

$$K \subset\subset \{x \in \mathbb{R}^k : \phi(x) = 1\} \text{ and } \mathbb{Z}^k \cap \text{supp}(\phi) = \mathbb{Z}^k \cap K.$$

Given f as in (B), we note that

$$h(w, x) = (1 - \phi(x))f(w, x) \text{ if } x \notin K \text{ and } h(w, x) = 0 \text{ if } x \in K$$

defines h on all of $W \times \mathbb{R}^k$. Furthermore, if f satisfies the assumption of (C), so does h . Parts (B) and (C) have already been proved for h . We note that

$$\sum_{n \in \mathbb{Z}^k} h(w, x) z^n = \sum_{n \in \mathbb{Z}^k} f(w, n) z^n.$$

This finishes the proof of (B) and (C) in general. (A) is the special case of (B) when W is a point. \square

Remark 3.8. The sum of the series in thm. 3.7(A) is in fact C^∞ at $1 \neq z \in \mathbb{T}^k$. This can be seen by noting that, with m as in the above proof, the same argument shows that $\Delta_i^{m+d} P f|_{\mathbb{Z}^k} \in L^1(\mathbb{Z}^k)$ for every polynomial of degree at most d . We skip the details. The statement is contained in lemma 6.5.

Corollary 3.9. *With f as in theorem 1.1, the series $\sum_{n \in \mathbb{Z}^k} f(x + n) z^n$ is t -summable for every $x \in \mathbb{R}^k$ and $1 \neq z \in \mathbb{T}^k$.*

Proof. This is a special case of the above theorem, once it is noted that (i) homogeneity implies \mathcal{H}_∞ , and (ii) translates of \mathcal{H}_∞ functions are also \mathcal{H}_∞ . \square

Corollary 3.10. *Assume $f, g : \mathbb{R}^k \setminus \{0\} \rightarrow \mathbb{C}$ are both C^∞ and homogeneous of type $(1, 1)$ and (s_0, ϵ) respectively. Assume furthermore that $f(x)$ is a positive real number for all $0 \neq x \in \mathbb{R}^k$. Then the t -convergent sum $\sum_{n \in \mathbb{Z}^k} f(x)^{-s} g(x) z^n$ is a holomorphic function of $s \in \mathbb{C}$ for all $x \in \mathbb{R}^k$ and for all $1 \neq z \in \mathbb{T}^k$.*

Proof. To apply the above theorem, one has to observe that $h(s, x) = f(x)^{-s} g(x)$ is uniformly \mathcal{H}_∞ on $W \times \{x \in \mathbb{R}^k : \|x\| > r\}$ where W is a bounded subset of \mathbb{C} , and r is any positive real number. \square

As remarked in the introduction, this implies the analytic continuation of $L(\chi, s)$ for nontrivial Dirichlet characters χ .

The proofs of t -summability given so far suggest a broader definition, that of t -integrability on \mathbb{R}^k . This requires the language of distributions, and is given in 6.10.

4. h-SUMMABILITY ON \mathbb{Q}^k

Remark 4.1. Before beginning on h-summability, we make a simple observation on t-summability, which will be required in the sequel. Let Γ_1 be a subgroup of a commutative group Γ_2 . Let $a_1 : \Gamma_1 \rightarrow \mathbb{C}$ be any function, and extend it by zero to obtain $a_2 : \Gamma_2 \rightarrow \mathbb{C}$. Then a_1 is t-summable if and only if a_2 is t-summable.

Notation 4.2. Recall that h-summability has been defined for R -modules Γ , where R is a commutative ring. In this section, $R = \mathbb{Q}$ and $\Gamma = \mathbb{Q}^k$.

Given $a : \mathbb{Q}^k \rightarrow \mathbb{C}$ and a subset $M \subset \mathbb{Q}^k$, we have $R_M a : \mathbb{Q}^k \rightarrow \mathbb{C}$ given by

$$R_M a(x) = a(x) \text{ if } x \in M, \text{ and } R_M a(x) = 0 \text{ if } x \notin M.$$

For $t \in \mathbb{Q}^\times$ and $a : \mathbb{Q}^k \rightarrow \mathbb{C}$, we define $\rho(t)a(x) = a(t^{-1}x)$, $\forall x \in \mathbb{Q}^k$. We observe that

$$(17) \quad \rho(t)R_M \rho(t)^{-1}f = R_{tM}f \quad \forall M \subset \mathbb{Q}^k, \forall t \in \mathbb{Q}^\times, \forall f : \mathbb{Q}^k \rightarrow \mathbb{C}.$$

Given $M \subset \mathbb{Q}^k$ and $a : M \rightarrow \mathbb{C}$, we extend a by zero and obtain $a' : \mathbb{Q}^k \rightarrow \mathbb{C}$. When $\sum_{n \in \mathbb{Q}^k} a'(n)$ is h-summable, we will often say that $\sum_{n \in M} a(n)$ is h-summable, or even more simply that a is h-summable.

Proposition 4.3. *Let $f : \mathbb{R}^k \setminus \{0\} \rightarrow \mathbb{C}$ be C^∞ and homogeneous of type $(-s, \epsilon)$. Then*

- (1) $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$ is h-summable if $(-s, \epsilon) \neq (k, 1)$.
- (2) Assume $x \notin \mathbb{Z}^k$. Then $\sum_{n \in \mathbb{Z}^k} f(x+n)$ is h-summable if $(-s, \epsilon) \notin \{(k-i, (-1)^i) : i = 0, 1, 2, \dots\}$.

Proof. We define $f(0) = 0$ and regard \mathbb{R}^k as the domain of f .

$\sum_{n \in \mathbb{Z}^k} f(n)z^n$ is t-convergent for $z \neq 1$ by Corollary 3.9. Summing over the nontrivial N-torsion points of \mathbb{T}^k , we deduce that $N^k R_{N\mathbb{Z}^k} f - R_{\mathbb{Z}^k} f$ is t-summable. Thanks to the homogeneity assumption and (17), we get

$$(18) \quad (N^{k-s} \rho(N) - 1) R_{\mathbb{Z}^k} f \text{ is t-summable on } \mathbb{Q}^k.$$

It follows that $R_{\mathbb{Z}^k} f$ is h-summable if $(N^{k-s} - 1) \neq 0$, and that its sum is given by

$$(19) \quad \sum_{0 \neq n \in \mathbb{Z}^k} f(n) = (N^{k-s} - 1)^{-1} \sum \left\{ \sum_{0 \neq n \in \mathbb{Z}^k} f(n)z^n : 1 \neq z \in \mathbb{T}^k, z^N = 1 \right\}.$$

If $s \neq k$, one may choose such a natural number N . If $\epsilon = -1$, then $(1 + \rho(-1))R_{\mathbb{Z}^k} f = 0$ so it is h-summable in this case as well. This proves part (1).

Fix $x \in \mathbb{R}^k$. Let $f_x(v) = f(x+v)$ for every $v \in \mathbb{R}^k$. Utilizing the Taylor expansion of f at $v \in \mathbb{R}^k$ we obtain

$$f(v+x) = g_0(v) + g_1(v) + \dots + g_{m-1}(v) + \text{Rem}_m(v) \text{ when } \|v\| > \|x\|$$

We note that $g_i(v) = \partial_x^i f(v)/i!$ is C^∞ on $\mathbb{R}^k \setminus \{0\}$ and homogeneous of type $(-(s+i), (-1)^i \epsilon)$. By part (1) we see that $R_{\mathbb{Z}^k} g_i$ is h-convergent. If $m + \text{Re}(s) = h > k$ we see that $\text{Rem}_m(v)$ is $O(\|v\|^{-h})$ as $\|v\| \rightarrow \infty$, and so $\text{Rem}_m|_{\mathbb{Z}^k}$ is in $L^1(\mathbb{Z}^k)$. It follows that $R_{\mathbb{Z}^k} f_x$, being a finite sum of h-convergent series, is itself h-convergent.

□

Lemma 4.4. *If $f : \mathbb{R}^k \setminus \{0\} \rightarrow \mathbb{C}$ is C^1 , homogeneous of type $(-k, 1)$, and if $\sum_{0 \neq n \in \mathbb{Z}^k} f(n)$ is h-summable, then $f = 0$.*

Proof. We first assume that f is C^∞ . We set $f(0) = 0$. Recall the subspaces

$$L_t^1(\mathbb{Q}^k) \subset L_h^1(\mathbb{Q}^k) \subset \mathbb{C}^{\mathbb{Q}^k} = V$$

as in Proposition 2.2. From (18), we see that $(\rho(N) - 1)R_{\mathbb{Z}^k}f \in L_t^1(\mathbb{Q}^k)$ for every natural number N . In addition we have $\rho(-1)R_{\mathbb{Z}^k}f = R_{\mathbb{Z}^k}f$. It follows that

(i) the image of $R_{\mathbb{Z}^k}f$ in $V/L_t^1(\mathbb{Q}^k)$ is invariant under the action of $\rho(\mathbb{Q}^\times)$.

From (i) and the definition of h-summability, we get

(ii) $R_{\mathbb{Z}^k}f$ is h-summable if and only if it is t-summable.

By remark 4.1 with $\Gamma_1 = \mathbb{Z}^k$ and $\Gamma_2 = \mathbb{Q}^k$, we note:

(iii) $R_{\mathbb{Z}^k}f$ is t-summable if and only if $f|_{\mathbb{Z}^k} : \mathbb{Z}^k \rightarrow \mathbb{C}$ is t-summable.

For every $v \in \mathbb{Z}^k$, we observe that $n \mapsto f(n+v) - f(n)$ gives an element of $L^1(\mathbb{Z}^k)$. It follows that $f|_{\mathbb{Z}^k}$ is t-summable if and only if it belongs to $L^1(\mathbb{Z}^k)$. Comparing the integral and the sum on conical regions, we see that if $f|_{\mathbb{Z}^k}$ is in $L^1(\mathbb{Z}^k)$, then f is identically zero.

The reader is left to check that the C^1 hypothesis is sufficient for the validity of the above argument. □

Proposition 4.5. *If f is a polynomial, the series $\sum_{n \in \mathbb{Z}^k} f(n)z^n$ is h-summable for all $z \in \mathbb{T}^k$, and the sum of this series is zero.*

Proof. By lemma 3.4 we see that this series is t-summable and that its sum is zero for $1 \neq z \in \mathbb{T}^k$. For $z = 1$ and f homogeneous, the h-summability is contained in prop. 4.3, where it is also shown that the sum of this series is a linear combination of $\sum_{n \in \mathbb{Z}^k} f(n)z^n$ for certain $z \neq 1$, and is thus equal to zero. Linearity proves the result for all polynomials. □

Proposition 4.6. *Let f, g be as in corollary 3.10, with $\epsilon = 1$. Then the h-convergent sum $h(s) = \sum_{0 \neq n \in \mathbb{Z}^k} f(n)^{-s} g(n)$ is a holomorphic function of $s \neq (k + s_0)$. Furthermore, h has a simple pole at worst at $(k + s_0)$.*

Proof. Both assertions follow from equation (19) for $f^{-s}g$. The t-convergent sums on the right side of that equation are holomorphic by Corollary 3.10, and $(N^{k+s_0-s} - 1)$ has a simple zero at $s = k + s_0$. □

We discuss next an inhomogeneous situation: express $P \in \mathbb{R}[x_1, x_2, \dots, x_k]$ as the sum of homogeneous polynomials: $P = P_0 + P_1 + \dots + P_d$. Assume that $P_d(x) > 0$ for all nonzero $x \in \mathbb{R}^k$. Evidently, $K = \{x \in \mathbb{R}^k : P(x) \leq 0\}$ is compact. Let $F \subset \mathbb{Z}^k$ be any finite subset that contains $K \cap \mathbb{Z}^k$.

Theorem 4.7. *With P and F as above, let $G(s)$ denote the sum of the series $\sum_{n \in \mathbb{Z}^k \setminus F} P(x)^{-s}$ whenever this series is h-convergent. Let $E = \{(k - j)/d : j \in 2\mathbb{Z}, j \geq 0\}$. Then*

- (1) $G(s)$ is defined for all $s \notin E$, and $G|_{\mathbb{C} \setminus E}$ is holomorphic. This function has meromorphic continuation with simple poles, denoted by $\tilde{G}(s)$, to all of \mathbb{C} .
- (2) Let $s_0 \in E$. If $G(s_0)$ is defined, then \tilde{G} is holomorphic at s_0 . But $\tilde{G}(s) \neq G(s)$ in general.
- (3) If $s \in \mathbb{Z}, s \leq 0$, then $G(s)$ is defined and equals $-\Sigma\{P(n)^{-s} : n \in F\}$. Therefore \tilde{G} is holomorphic at all such s .
- (4) If k is odd, then $\tilde{G}(s) = -\Sigma\{P(n)^{-s} : n \in F\}$ when $s \in \mathbb{Z}, s \leq 0$.

Remark 4.8. The requirement that k is odd in part (4) can be altered in the following manner. We consider instead the h-convergence of $Q.P^{-s}|_{\mathbb{Z}^k \setminus F}$ where Q is a homogeneous polynomial of degree e and once again denote by \tilde{G} the sum of the series extended meromorphically. If $s \in \mathbb{Z}$ and $s \leq 0$, then $\tilde{G}(s)$ is once again the negative of the sum $Q(n)P(n)^{-s}$ taken over $n \in F$, under the assumption that $(e+k)$ is odd. The proof of this is a notational modification of the proof of the theorem given below.

Proof. The statement of the theorem remains unaffected if F is replaced by a larger finite subset. So we will assume that $0 \in F$.

Write $P = P_d(1 - R)$, then $R(x)$ is $O(\|x\|^{-1})$ as $\|x\| \rightarrow \infty$, so we may approximate $(1 - R)^{-s}$ through the power series expansion:

$$(20) \quad (1 - t)^{-s} = \sum_{e=0}^{\infty} a_e(s)t^e, A_m(s, t) = \sum_{e=0}^{e=m-1} a_e(s)t^e, E_m(s, t) = (1 - t)^{-s} - A_m(s, t)$$

noting that $|E_m(s, t)| \leq C(m, r)|t|^m$ if $|s| < r$ and $|t| \leq 1/2$. We deduce bounds $|E_m(s, R(x))| \leq C'(m, r)\|x\|^{-m}$ valid when $|s| < r$ and $\|x\| > M(m, r)$. Choose m so that $m + d\operatorname{Re}(s) > k$. Let $W = \{s \in \mathbb{C} : |s| < r\}$. For every $s \in W$, we see that $x \mapsto P_d(x)^{-s}E_m(s, x)$ for $x \in \mathbb{Z}^k \setminus F$ is in $L^1(\mathbb{Z}^k \setminus F)$, and furthermore, these members of $L^1(\mathbb{Z}^k \setminus F)$ lie in a bounded set. By remark 3.5, we see that

$$(21) \quad s \mapsto e_m(s) = \sum_{n \in \mathbb{Z}^k \setminus F} P_d(n)^{-s} E_m(s, R(n)) \text{ is holomorphic on } |s| < r.$$

We note that R belongs to the \mathbb{Z} -graded ring obtained by adjoining P_d^{-1} to $\mathbb{R}[x_1, x_2, \dots, x_k]$. We write $R^i = (R^i)_i + (R^i)_{i+1} \dots + (R^i)_{id}$ where each $(R^i)_j$ is homogeneous of degree $(-j)$. Putting $t = R(x)$ in (20) and multiplying by $P_d(x)^{-s}$ we get

$$(22) \quad P(x)^{-s} = P_d(x)^{-s} E_m(s, x) + \sum_j P_d(x)^{-s} B_j(s, x) \text{ with } B_j(s, x) = \sum_e a_e(s) (R^e)_j.$$

By Proposition 4.3, the term $P_d(x)^{-s} (R^e)_j$, being homogeneous of degree $-(ds + j)$, $(-1)^j$, is h-summable when restricted to $\mathbb{Z}^k \setminus F$, unless $ds + j = k$ and $j \in 2\mathbb{Z}$. Thus it is h-summable if $s \notin E$; denote its sum by $g_{e,j}(s)$. By (22), we see that the series in question has been expressed as a linear combination of h-summable series. So we conclude

$$(23) \quad \text{If } |s| < r, s \notin E, \text{ then } G(s) \text{ is defined and equals } e_m(s) + \sum_j \sum_e a_e(s) g_{e,j}(s).$$

Prop. 4.6(1), and (21) with arbitrarily large r , combine to prove part (1).

Next, we take $s_0 \in E$ and then study (i) the behaviour of \tilde{G} on the region $U = \{s : |s - s_0| < 2/d\}$, and (ii) the h-summability of $P^{-s_0}|_{\mathbb{Z}^k \setminus F}$. By assumption, we have $p \in 2\mathbb{Z}, p \geq 0$ and $ds_0 + p = k$. For every $s \in U$, the restrictions of all the functions

listed in (22), with the exception of $P_d(x)^{-s}B_p(s, x)$, when restricted to $\mathbb{Z}^k \setminus \{0\}$ are h-summable. Furthermore, their sums are holomorphic functions on U . It only remains to consider $B_p(s, x)$, which it is more convenient to express as the finite sum:

$$B_p(s, x) = V_0(x) + (s - s_0)V_1(x) + \dots + (s - s_0)^q V_q(x)$$

where all the $V_i(x) \in \mathbb{R}[x_1, \dots, x_k]_{P_d}$ are homogeneous of degree $(-i)$. Let $L_i(s) = \sum_{n \in \mathbb{Z}^k \setminus F} P_d(n)^{-s} V_i(n)$ for all $s \neq s_0$. By 4.6, we see that $(s - s_0)^i L_i(s)$ is holomorphic on U for all $i > 0$. Putting this together, we see:

- (A) $\tilde{G} - L_0$ is holomorphic on U , and
- (B) $\sum_{n \in \mathbb{Z}^k \setminus F} P(n)^{-s_0} - P_d^{-s_0}(n)V_0(n)$ is h-summable.

Now assume that the given series is h-summable at s_0 . It follows from (B) that $P_d^{-s_0}V_0|_{\mathbb{Z}^k \setminus F}$ is also h-summable. By (4.4), we see that $P_d^{-s_0}V_0 = 0$. It follows that $P_d^{-s}V_0 = 0$ and therefore L_0 vanishes as well. We conclude from (A) that \tilde{G} is holomorphic on U . We see however that $\tilde{G}(s_0) = G(s_0) + b$ where b is the residue of $L_1(s)$ at s_0 . If $k = 2$, $P(x, y) = x^2 + y^2 - x$, $s_0 = 0$, one checks that b is precisely the residue of $\zeta_K(s)$ at $s = 1$, where $K = \mathbb{Q}(\sqrt{-1})$, which is $\pi/4$. This completes the proof of part (2).

Proposition 4.5 implies part (3).

Note that $d \in 2\mathbb{Z}$ and therefore $\mathbb{Z} \cap E$ is empty when k is odd. Part (4) is now implied by parts (1) and (3). This completes the proof of the theorem. \square

5. THE FORMALISM OF THE POISSON FORMULA

A nice account of the Poisson formula is to be found in the books of Lang and Weil, [7] and [15]. A sketch is given in the beginning of this section. This formula, for the original function replaced by a translate and then multiplied by a character, occurs in (36). The view of this formula taken here is borrowed from [16],[9] and [10]. A statement equivalent to the Poisson formula is Prop.5.2(2). Here the Fourier transform is expressed as the composite of linear operators that are defined on Frechet spaces, such as the Schwartz space $\mathcal{S}(X)$ of C^∞ functions of rapid decay on X and $C^\infty(X \times X'/\Gamma \times \Gamma')$ defined below. The operators that appear in Prop. 5.2 preserve inner products. Thus they extend to the complex conjugates of their dual spaces; this is Prop.5.4. The preceding remark 5.3 identifies the latter objects with spaces of distributions of a certain type. While Prop. 5.4 extends operators such as T_B given in (30), it treats Σ_Γ as a packet, and does not permit one to separate the individual terms of the series indexed by $\gamma \in \Gamma$. We think of Theorem 5.8(5) as the ‘true’ Poisson formula for tempered distributions, and that is first objective of this section.

We then proceed to sieve out the obvious contributions to the singularities of $T_B u$ at the origin to obtain $T_B^{reg} u$ in definition 5.10. The next objective is to give sufficient conditions that ensure $T_B^{reg} u$ is C^∞ in a neighborhood of zero (see Proposition 5.11).

Except for the factor 2π in the Fourier transform, the notation here for distributions, and operations on distributions, is completely consistent with that of Hormander’s book [5]. The facts on distributions that we use are seen in a first course on the subject: they are contained in vol.1, chapter 7, [5], and also to be

found in chapters 6,7 of [14].

Our data consists of a four-tuple (X, X', B, Γ) where X and X' are finite dimensional \mathbb{R} -vector spaces, $B : X \times X' \rightarrow \mathbb{R}$ is a non-degenerate bilinear form, and $\Gamma \subset X$ is a lattice, i.e. Γ is discrete and X/Γ is compact. Every (X, X', B, Γ) as above produces its *dual* (X', X, B', Γ') given by

$$(24) \quad B'(x', x) = -B(x, x') \forall x \in X, x' \in X', \quad \Gamma' = \{\gamma' \in \Gamma' : B(\gamma, \gamma') \in \mathbb{Z} \forall \gamma \in \Gamma\}$$

The compact torus Z is defined by

$$(25) \quad Z = X \times X' / \Gamma \times \Gamma'$$

The Haar measure on X is chosen so that $\text{vol}(X/\Gamma) = 1$. The integral of a function with respect to this Haar measure will be denoted by $\int_X f(x)dx$ or even simply by $\int_X f$. We put

$$(26) \quad \psi(t) = \exp(2\pi\sqrt{-1}t) \forall t \in \mathbb{R} \text{ and } \psi_B(x, x') = \psi(B(x, x')) \forall x \in X, x' \in X'.$$

We recall that the Schwartz space of X , denoted by $\mathcal{S}(X)$, is the collection of $C^\infty \mathbb{C}$ -valued functions f defined on X for which $\|f\|_{(M,N)} < \infty$ for all non-negative integers M and N , where

$$(27) \quad \|f\|_{(M,N)} = \sup\{(1 + \|x\|)^N \|v\|^{-m} |\partial_v^m f(x)| : x \in X, 0 \neq v \in X, 0 \leq m \leq M\}$$

In the above, ∂_v denotes the directional derivative. Norms on both X and X' are chosen arbitrarily and fixed once and for all. The above semi-norms $\|\cdot\|_{(M,N)}$ give $\mathcal{S}(X)$ the structure of a topological vector space.

For all $f \in \mathcal{S}(X)$, its Fourier transform $\mathcal{F}_B f$ is the function on X' defined by the absolutely convergent integral

$$(28) \quad \mathcal{F}_B(f)(x') = \int_X f(x) \psi_B(x, x') dx.$$

If $f \in \mathcal{S}(X)$, then $\mathcal{F}_B f \in \mathcal{S}(X')$, and in fact $\mathcal{F}_B : \mathcal{S}(X) \rightarrow \mathcal{S}(X')$ is continuous. This statement follows from the standard identities below, valid for all $u \in \mathcal{S}(X)$

$$(29) \quad \partial_{x'} \mathcal{F}_B u = 2\pi\sqrt{-1} \mathcal{F}_B B_{x'} u \text{ and } \mathcal{F}_B \partial_x u = 2\pi\sqrt{-1} B'_x \mathcal{F}_B u$$

where $B_{x'}(x) = B(x, x')$ and $B'_x(x') = B'(x', x)$ for all $x \in X, x' \in X'$.

Let us return to (28), the Fourier integral. This integral may be computed, first by summing over $x + \Gamma$, and then integrating the resulting function on X/Γ . In the summation over $x + \Gamma$, there is a common factor $\psi_B(x, x')$ which we suppress for the moment. Thus, for $f \in \mathcal{S}(X)$, we define $T_B f : X \times X' \rightarrow \mathbb{C}$ by the formula

$$(30) \quad T_B f(x, x') = \sum_{\gamma \in \Gamma} f(x + \gamma) \psi_B(\gamma, x'),$$

and obtain the Fourier transform of $f \in \mathcal{S}(X)$ as

$$(31) \quad \mathcal{F}_B f(x') = \int_{X/\Gamma} T_B f(x, x') \psi_B(x, x') dx.$$

We note that the property below

$$(32) \quad u \text{ and } \psi_B.u \text{ are translation invariant by } \Gamma' \text{ and } \Gamma \text{ respectively,}$$

is valid for $u = T_B f$ for all $f \in \mathcal{S}(X)$. We define the space

$$(33) \quad C^\infty(X \times X'/\Gamma \times \Gamma')$$

to be collection of infinitely differentiable functions on $X \times X'$ that satisfies (32). For $f, g \in C^\infty(X \times X'/\Gamma \times \Gamma')$, the function $f.\bar{g} : X \times X' \rightarrow \mathbb{C}$ descends to a function $(f.\bar{g})_d$ on the torus Z defined in (25). We define

$$(34) \quad \langle f, g \rangle = \int_Z (f.\bar{g})_d$$

The theory of Fourier series for C^∞ functions on X'/Γ' suffices to deduce the statement below.

Proposition 5.1. (1) $T_B : \mathcal{S}(X) \rightarrow C^\infty(X \times X'/\Gamma \times \Gamma')$ is an isomorphism,
 (2) $\langle f, g \rangle = \langle T_B f, T_B g \rangle$ for all $f, g \in \mathcal{S}(X)$,
 (3) $S_B : C^\infty(X \times X'/\Gamma \times \Gamma') \rightarrow \mathcal{S}(X)$ given by $S_B h(x) = \int_{X'/\Gamma'} h(x, x') dx'$ is the inverse of T_B .

The operator in (30) for the dual (X', X, B', Γ') is denoted by $T_{B'}$ and the space given in (33) for the dual is denoted by $C^\infty(X' \times X/\Gamma' \times \Gamma)$. We define $\sigma_B : C^\infty(X \times X'/\Gamma \times \Gamma') \rightarrow C^\infty(X' \times X/\Gamma' \times \Gamma)$ by

$$(35) \quad (\sigma_B h)(x', x) = \psi_B(x, x') h(x, x') \quad \forall h \in C^\infty(X \times X'/\Gamma \times \Gamma')$$

The same considerations applied to the dual (X', X, B', Γ') give the additional topological vector space $C^\infty(X' \times X/\Gamma' \times \Gamma)$, and also the operators $T_{B'}, S_{B'}, \sigma_{B'}, \mathcal{F}_{B'}$.

The negative sign in the definition of B' , (31) and the proposition 5.1 now combine to give the statement below.

Proposition 5.2. (1) The operators $\sigma_B, T_B, T_{B'}$ are isomorphisms of topological vector spaces, and their inverses are $\sigma_{B'}, S_B, S_{B'}$ respectively.
 (2) $\mathcal{F}_B = T_{B'}^{-1} \circ \sigma_B \circ T_B$ and $\mathcal{F}_{B'} = T_B^{-1} \circ \sigma_{B'} \circ T_{B'}$, and thus \mathcal{F}_B and $\mathcal{F}_{B'}$ are inverses of each other.
 (3) All the operators listed above preserve inner products.

Part (2) of Proposition 5.2 shows $T_B f = \sigma_{B'} T_{B'} \mathcal{F}_{B'} f$ and this reads as:

$$(36) \quad \sum_{\gamma \in \Gamma} f(x + \gamma) \psi_B(\gamma, x') = \sum_{g' \in \Gamma'} \mathcal{F}_B(f)(x' + g') \psi_{B'}(x' + g', x) \quad \forall f \in \mathcal{S}(X)$$

The standard form of the Poisson formula is obtained by putting $(x, x') = (0, 0)$.

Remark 5.3. The space of distributions (resp. tempered distributions) on a finite dimensional real vector space V is denoted by $\mathcal{D}(V)$ (resp. $\mathcal{S}(V)^*$).

(A) We shall define $\langle, \rangle : \mathcal{S}(X)^* \times \mathcal{S}(X) \rightarrow \mathbb{C}$ by

$$\langle u, f \rangle = u(\bar{f}) \text{ for all } u \in \mathcal{S}(X)^*, f \in \mathcal{S}(X).$$

(i) The restriction of \langle, \rangle to $\mathcal{S}(X) \times \mathcal{S}(X)$ is the standard inner product.

(ii) Every continuous linear functional on $\mathcal{S}(X)$ is given by $f \mapsto \overline{\langle u, f \rangle}$ for a unique $u \in \mathcal{S}(X)^*$.

(B) Let $\mathcal{D}(X \times X'/\Gamma \times \Gamma')$ be the space of distributions u on $X \times X'$ that satisfy (32). We will define

$$\langle, \rangle : \mathcal{D}(X \times X'/\Gamma \times \Gamma') \times C^\infty(X \times X'/\Gamma \times \Gamma') \rightarrow \mathbb{C}$$

as follows. For $u \in \mathcal{D}(X \times X'/\Gamma \times \Gamma')$, $h \in C^\infty(X \times X'/\Gamma \times \Gamma')$, the distribution $\bar{h}.u$ is invariant under translation by $\Gamma \times \Gamma'$ and therefore descends to a distribution $(\bar{h}.u)_d$ on the torus Z defined in (25). Denoting by 1_Z the constant function 1 on Z , we define $\langle u, f \rangle = (\bar{h}.u)_d 1_Z$.

(i) If u is also in $C^\infty(X \times X'/\Gamma \times \Gamma')$, then this definition of $\langle u, h \rangle$ agrees with the formula of (34).

(ii) Every continuous linear functional on $C^\infty(X \times X'/\Gamma \times \Gamma')$ is given by $h \mapsto \overline{\langle u, h \rangle}$ for a unique $u \in \mathcal{D}(X \times X'/\Gamma \times \Gamma')$. One sees this by identifying $C^\infty(X \times X'/\Gamma \times \Gamma')$ with the global C^∞ sections of a unitary line bundle L on Z . The compactness of Z then gives the identification of $\overline{C^\infty(X \times X'/\Gamma \times \Gamma')^*}$ with the space of global sections of $L \otimes D$ where D is the sheaf of distributions on Z . The latter space is canonically identified with $\mathcal{D}(X \times X'/\Gamma \times \Gamma')$.

We also have (A) and (B) above for the dual (X', X, B', Γ') , namely

(A') $\langle, \rangle : \mathcal{S}(X')^* \times \mathcal{S}(X') \rightarrow \mathbb{C}$ and

(B') $\langle, \rangle : \mathcal{D}(X' \times X/\Gamma' \times \Gamma) \times C^\infty(X' \times X/\Gamma' \times \Gamma) \rightarrow \mathbb{C}$.

For $F = \mathcal{S}(X), \mathcal{S}(X'), C^\infty(X \times X'/\Gamma \times \Gamma'), C^\infty(X' \times X/\Gamma' \times \Gamma)$,

let $F_e = \mathcal{S}(X)^*, \mathcal{S}(X')^*, \mathcal{D}(X \times X'/\Gamma \times \Gamma'), \mathcal{D}(X' \times X/\Gamma' \times \Gamma)$ respectively.

Proposition 5.4. *Every operator $U : F_1 \rightarrow F_2$ that occurs in Prop. 5.2 extends to an isomorphism $(F_1)_e \rightarrow (F_2)_e$, once again denoted by U by abuse of notation, that is specified uniquely by*

$$(37) \quad \langle Uu, Uh \rangle = \langle u, h \rangle \text{ for all } u \in (F_1)_e, h \in F_1$$

Furthermore, all the identities enumerated in Prop. 5.2 are valid for the extended operators.

Proof. Properties (i) and (ii) of $\langle, \rangle : F_e \times F \rightarrow \mathbb{C}$ listed above, combined with the last assertion of Prop. 5.2, prove this statement. \square

By the above Proposition, $T_B u$ has been defined for all $u \in \mathcal{S}(X)^*$. Our next goal, theorem 5.8 below, is to prove the validity of (30) for $u \in \mathcal{S}(X)^*$. For this purpose, we first give meaning to the γ -th term in (30) for every distribution u on X in the standard manner.

In order to obtain the formula

$$(38) \quad \int_{X \times X'} f(x + \gamma) \psi_B(\gamma, x') \phi(x, x') dx dx' = \int_X f(x) I_\gamma \phi(x) dx$$

for all $\gamma \in \Gamma$, $f \in \mathcal{S}(X)$ and all test functions $\phi \in C_c^\infty(X \times X')$, we define $I_\gamma \phi \in C_c^\infty(X)$ by

$$(39) \quad I_\gamma \phi(x) = \int_{X'} \psi_B(\gamma, x') \phi(x - \gamma, x') dx'.$$

For a distribution u on X we define the distribution u_γ on $X \times X'$ by

$$(40) \quad u_\gamma \phi = u(I_\gamma \phi) \text{ } \forall \text{ test functions } \phi \in C_c^\infty(X \times X').$$

By (38), we see that if $u = f \in \mathcal{S}(X)$, then u_γ is given by the function $(x, x') \mapsto f(x + \gamma)\psi_B(\gamma, x')$, as desired. Lemma 5.5 is required to show that $\sum_{\gamma \in \Gamma} u_\gamma$ converges to a distribution on $X \times X'$.

For $\phi \in C_c^\infty(X \times X')$ we define

$$\|\phi\|^{(M, N)} = \sup\{\|x\|^{-m}\|x'\|^{-n}\|\partial_x^m \partial_{x'}^n \phi\|_\infty : 0 \neq x \in X, 0 \neq x' \in X', 0 \leq m \leq M, 0 \leq n \leq N\}$$

Lemma 5.5. *Let $K \subset X$ and $K' \subset X'$ be compact subsets. Then, for every $M, N \geq 0$, there is a constant $C(K, K', M, N)$ with the property that the inequality*

$$(41) \quad \sum_{\gamma \in \Gamma} \|I_\gamma \phi\|_{(M, N)} \leq C(K, K', M, N) \|\phi\|^{(M, N+a)}$$

holds for every $\phi \in C_c^\infty(K \times K')$, with $a = 1 + \dim X$ and notation as in (27).

Proof. Putting $L_x(x') = \phi(x, x')$, we first note that $I_\gamma \phi(x) = \mathcal{F}_{B'} L_{x-\gamma}(-\gamma)$ for all $x \in X, \gamma \in \Gamma$.

$\int_{X'} (1 + \|x'\|)^{-a} < \infty$ and the standard identities (29) imply

$$(42) \quad \|\mathcal{F}_{B'} f'\|_{(0, N)} \leq C_1(N) \|f'\|_{(N, a)} \forall f' \in \mathcal{S}(X')$$

for every N . Let $b = 1 + \sup\{\|x'\| : x' \in K'\}$. We may now rewrite the above inequality for $f' = L_x$ in the form below:

$$(43) \quad (1 + \|y\|)^N |\mathcal{F}_{B'} L_x(y)| \leq b^a C_1(N) \|\phi\|^{(0, N)} \text{ for all } x, y \in X$$

From the compactness of K , we get $c > 1$ for which

$$(44) \quad c^{-1}(1 + \|y - x\|) \leq (1 + \|y\|) \leq c(1 + \|y - x\|) \text{ for all } x \in K, y \in X.$$

Replacing N by $N + a$ in (43), we may now rewrite that inequality in the form

$$(45) \quad (1 + \|y\|)^a (1 + \|y - x\|)^N |\mathcal{F}_{B'} L_x(y)| \leq b^a c^a C_1(N + a) \|\phi\|^{(0, N+a)} \text{ for all } x, y \in X.$$

Replacing (x, y) in the above by $(x - \gamma, -\gamma)$ we get

$$(46) \quad (1 + \|\gamma\|)^a (1 + \|x\|)^N |I_\gamma \phi(x)| \leq b^a c^a C_1(N + a) \|\phi\|^{(0, N+a)} \text{ for all } x \in X, \gamma \in \Gamma.$$

Taking $C(K, K', 0, N) = b^a c^a C_1(N + a) \sum_{\gamma \in \Gamma} (1 + \|\gamma\|)^{-a}$, we have proved (41) when $M = 0$.

We note that $I_\gamma \partial_x \phi = \partial_x I_\gamma \phi$ for all $x \in X$. We then see that the inequality (41) for $(0, N)$ applied to all the partial derivatives of ϕ of order at most M proves (41) for (M, N) as well. □

Lemma 5.6. *Let $\phi \in C_c^\infty(X \times X')$. Then $W\phi$ given by*

$$W\phi(x, x') = \sum_{\gamma \in \Gamma} \sum_{\gamma' \in \Gamma'} \phi(x + \gamma, x' + \gamma') \psi_B(\gamma, x')$$

has the properties below

- (1) $W\phi \in C^\infty(X \times X' / \Gamma \times \Gamma')$
- (2) $u(\bar{\phi}) = \langle u, W\phi \rangle$ for all $u \in \mathcal{D}(X \times X' / \Gamma \times \Gamma')$ with \langle, \rangle as given in remark 5.3(B)
- (3) $W : C_c^\infty(X \times X') \rightarrow C^\infty(X \times X' / \Gamma \times \Gamma')$ is surjective.

Proof. Let $\Omega \subset X \times X'$ be an open subset whose translates by all $(\gamma, \gamma') \in \Gamma \times \Gamma'$ are all disjoint. With Z as in (25), let $P : X \times X' \rightarrow Z$ denote the projection. Let ϕ be a test function with $\text{supp}(\phi) \subset \Omega$. We observe

$$(47) \quad h \in C^\infty(X \times X' / \Gamma \times \Gamma'), \quad h|_\Omega = \phi|_\Omega, \quad \text{supp}(h) \subset P^{-1}P\Omega \iff h = W\phi.$$

Let $u \in \mathcal{D}(X \times X' / \Gamma \times \Gamma')$. Recall that $\overline{W\phi}.u$ descends to a distribution $(\overline{W\phi}.u)_d$ on Z , and that $(\overline{W\phi}.u)_d 1_Z$ is the definition of $\langle u, W\phi \rangle$. For this purpose, the constant function 1_Z may be replaced by any smooth function $g : Z \rightarrow \mathbb{C}$ so that $g(z) = 1$ on an open subset of Z that contains $P\text{supp}(\phi)$. To obtain such a g , we choose another test function ϕ' on $X \times X'$ satisfying

$$\text{supp}(\phi') \subset \Omega \text{ and } \text{supp}(\phi) \subset (\phi')^{-1}\{1\}$$

and let $g : Z \rightarrow \mathbb{C}$ be the unique function with $\text{supp}(g) \subset P(\Omega)$ and $\phi'(z) = g(P(z))$ for all $z \in \Omega$. We then have

$$\langle u, W\phi \rangle = (\overline{W\phi}.u)_d g = (\overline{\phi}.u)(\phi') = u(\overline{\phi}.\phi') = u(\overline{\phi})$$

This proves the first two assertions for such ϕ . The linear span of $C_c^\infty(\Omega)$, taken over all Ω as above, is the collection of all test functions. Thus the first two assertions of the lemma follow from linearity. The same reasoning, combined with (47), proves the third assertion as well. \square

Lemma 5.7. *Let $J_\gamma \overline{\phi} = \overline{I_\gamma \phi}$ for all $\phi \in C_c^\infty(X \times X'), \gamma \in \Gamma$. For every $x \in X$, the finite sum $\sum_{\gamma \in \Gamma} J_\gamma(x)$ equals $\int_{X'/\Gamma'} W(x, x')$.*

Proof. Now $J_\gamma \phi(x) = \int_{X'} \psi_B(-\gamma, x') \phi(x - \gamma, x') dx'$. Once again, this integral may be computed by first summing over $x' + \Gamma'$ and then integrating the resulting function on X'/Γ' . So we get:

$$(48) \quad A_\gamma \phi(x, x') = \sum_{\gamma' \in \Gamma'} \psi_B(-\gamma, x') \phi(x - \gamma, x' + \gamma') \text{ and } J_\gamma \phi(x) = \int_{X'/\Gamma'} A_\gamma \phi(x, x')$$

and now summing the above over $\gamma \in \Gamma$, we get

$$(49) \quad W\phi(x, x') = \sum_{\gamma \in \Gamma} A_\gamma \phi(x, x') \text{ and } \sum_{\gamma \in \Gamma} J_\gamma \phi(x) = \int_{X'/\Gamma'} W\phi(x, x').$$

\square

Theorem 5.8. *Let u be a tempered distribution on X .*

- (1) *Let S be a subset of Γ . For every $\phi \in C_c^\infty(X \times X')$, the series $\sum_{\gamma \in S} u_\gamma \phi$ converges absolutely (see (40) for the definition of u_γ).*
- (2) *Denote the sum of the above series by $u_S \phi$. Then $\phi \mapsto u_S \phi$ defines a distribution on $X \times X'$.*
- (3) *If S is the disjoint union of subsets $S', S'' \subset \Gamma$, then $u_S = u_{S'} + u_{S''}$.*
- (4) *The distribution $T_B u$ given by Proposition 5.4 equals the above u_S when $S = \Gamma$.*
- (5) *$u_\Gamma = \sigma_{B'}(\mathcal{F}_B u)_{\Gamma'}$.*

Proof. Let $u \in \mathcal{S}(X)^*$. By definition, there are some M, N, C so that $|u(f)| \leq C \|f\|_{(M, N)}$. Recall that $u_\gamma \phi = u(I_\gamma \phi)$ for all $\gamma \in \Gamma$. Lemma 5.5 now shows that

$$\sum_{\gamma \in S} |u(I_\gamma \phi)| \leq C.C(K, K', M, N) \|\phi\|^{(M, N+a)} \forall \phi \in C_c^\infty(K \times K').$$

That proves part (1). The same upper bound holds for $|u_S\phi|$ as well, and now part (2) follows from the definition of a distribution. Part (3) is evident.

We now address part (4). It is clear that u_Γ is in $\mathcal{D}(X \times X' / \Gamma \times \Gamma')$. By Proposition 5.4, the identity $T_B u = u_\Gamma$ is equivalent to the equality $\langle u_\Gamma, T_B f \rangle = \langle u, f \rangle$ for all $f \in \mathcal{S}(X)$. Now S_B in Proposition 5.2 is surjective, and so is W (see lemma 5.6(3)). So it suffices to check this equality for $f = S_B W \phi$ for all $\phi \in C_c^\infty(X \times X')$. In other words, we have to check

$$\langle u_\Gamma, W \phi \rangle = \langle u, S_B W \phi \rangle \text{ for all } \phi \in C_c^\infty(X \times X')$$

We note

$$\langle u_\Gamma, W \phi \rangle = u_\Gamma \bar{\phi} = u \left(\sum_{\gamma \in \Gamma} I_\gamma \bar{\phi} \right) = u \left(\sum_{\gamma \in \Gamma} \overline{J_\gamma \phi} \right) = u \int_{X'/\Gamma'} \overline{W \phi} = u(\overline{S_B W \phi}) = \langle u, S_B W \phi \rangle$$

from 5.6(2) and 5.7, and this completes the proof of part (4).

The validity of part (4) for both (X, X', B, Γ) and (X', X, B', Γ') , combined with Proposition 5.4 now implies part (5), and therefore completes the proof of the theorem. \square

Lemma 5.9. *Let $u \in \mathcal{S}(X)^*$. Let $U \subset X$ be an open subset so that the restriction $u|_U$ is given by a continuous function $f : U \rightarrow \mathbb{C}$. Let A be a subset of Γ . Let $\Omega \subset X$ be an open subset so that $\Omega + A \subset U$. Assume further that $\sum_{\gamma \in A} \|f|_{\gamma+\Omega}\|_\infty < \infty$. Then the distribution u_A given in thm. 5.8 is given by the continuous function $x \mapsto \sum_{\gamma \in A} f(x + \gamma) \psi_B(\gamma, x')$ on $\Omega \times X'$.*

Proof. Let $\phi \in C_c^\infty(\Omega \times X')$ and let $\gamma \in A$. We observe that $I_\gamma \phi \in C_c^\infty(U)$ and also that equation (38) holds (in fact for any continuous function $f : U \rightarrow \mathbb{C}$). It then follows from the definition of u_γ given in (40) that

$$u_\gamma \phi = \int_{X \times X'} f(x + \gamma) \psi_B(\gamma, x') \phi(x, x') dx dx'.$$

Summing over $\gamma \in S$ we obtain

$$T_B^S u(\phi) = \int_{X \times X'} \left(\sum_{\gamma \in S} f(x + \gamma) \psi_B(\gamma, x') \right) \phi(x, x') dx dx'$$

for all test functions ϕ with support contained in $\Omega \times X'$. \square

In particular, taking $A = \{\gamma\}$, we see that if γ is not in the singular support of u , then $u_\gamma|_{U \times X'}$ is C^∞ for a suitable neighbourhood U of zero in X . This leads to the definition below.

Definition 5.10. Let $u \in \mathcal{S}(X)^*$. Let K and K' denote the singular supports of u and $\mathcal{F}_B u$. We assume that both K and K' are compact. Let

$$S = (\Gamma \cap K) \cup \{0\} \text{ and } S' = (\Gamma' \cap K') \cup \{0\}.$$

We define

$$T_B^{reg} u = T_B u - \sum_{\gamma \in S} u_\gamma - \sigma_{B'} \sum_{\gamma' \in S'} (\mathcal{F}_B u)_{\gamma'}$$

For the above formula, one should note that $u'_{\gamma'}$ and $u'_{S'}$ are defined for arbitrary $u' \in \mathcal{S}(X')^*$, $\gamma' \in \Gamma'$, $S' \subset \Gamma'$ by applying (40) and 5.8(2) to the dual (X', X, B', Γ') .

There is no real reason here to throw 0 into the definition of S and S' . This has been done for the sole purpose of obtaining consistency with the usage of $T^{reg}u$ in the next section.

Proposition 5.11. *Let $u = v + f + \mathcal{F}_{B'}v'$ where $f \in \mathcal{S}(X)$ and v and v' are compactly supported distributions on X and X' respectively. Then*

- (1) $T_B^{reg}u$ is defined, and is C^∞ on $U \times U'$ where U and U' are suitable neighbourhoods of 0 in X and X' respectively.
- (2) Let K, K', S, S' be as in definition 5.10. Then (1) holds with

$$X \setminus U = \{x - \gamma : x \in K, \gamma \in \Gamma \setminus S\} \text{ and } X' \setminus U' = \{x' - \gamma' : x' \in K', \gamma' \in \Gamma' \setminus S'\}.$$

Proof. Thanks to Payley-Weiner-Schwartz, we see that $\mathcal{F}_B v$ and $\mathcal{F}_{B'} v'$ are both C^∞ . It follows that the singular supports of u and $\mathcal{F}_B u$ are precisely the singular supports of v and v' respectively. These sets are compact, and therefore $T_B^{reg}u$ is defined. We retain the notation K, K', S, S' introduced in 5.10.

For $w \in \mathcal{S}(X)^*$, let

$$-Pw = \sum_{\gamma \in S} w_\gamma \text{ and } -Qw = \sigma_{B'} \sum_{\gamma' \in S'} (\mathcal{F}_B w)_{\gamma'}.$$

Now $T_B^{reg}u$ is the sum of nine terms, obtained by applying the three operators T_B, P, Q to the three distributions $v, f, \mathcal{F}_{B'}v'$.

Five of these, namely $T_B f, P f, Q f, Q v$ and $P \mathcal{F}_{B'} v'$, are evidently C^∞ on all of $X \times X'$. We claim that $(T_B + P)v, (T_B + Q)\mathcal{F}_{B'}v'$ are C^∞ on $U \times X'$ and $X \times U'$ respectively, where $U \subset X$ and $U' \subset X'$ are suitable neighborhoods of zero. This claim implies part (1) of the proposition.

To check this claim, let L be the support of v , and define the subsets $C, D \subset \Gamma$ by

$$\{0\} \cup (L \cap \Gamma) = S \sqcup C \text{ and } \Gamma = S \sqcup C \sqcup D.$$

By 5.8, we see that $T_B v = v_S + v_C + v_D$. If U is small enough, we see that $v_\gamma|U \times X'$ is

- (i) C^∞ if $\gamma \in C$, and
- (ii) is zero if $\gamma \in D$.

From the definition of v_D in 5.8 it follows that $v_D|U \times X'$ is zero. The finiteness of C shows that v_C is C^∞ on $U \times X'$. Now $Pv = -v_S$, so we see that $(T_B + P)v$ equals v_C which is C^∞ on $U \times X'$ as claimed.

To go further, we note that (i) and (ii) above are valid when U is the complement of the union of $R_1 = \{x - \gamma : x \in K, \gamma \in C\}$ and $R_2 = \{x - \gamma : x \in L, \gamma \in D\}$. We may express $v + f$ as $v_1 + f_1$ where $v_1 = \phi.v$ and $f_1 = f + (1 - \phi).v$ for a test function ϕ that is 1 on an open subset V that contains K . Now the support of v_1 is contained in V . It follows that we may replace L by K in the definition of R_2 . Because $S \sqcup C \sqcup D = \Gamma$, we deduce that U can be chosen to be the complement of $\{x - \gamma : x \in K, \gamma \in \Gamma, \gamma \notin S\}$.

With $r = T_{B'}v' - \sum_{\gamma' \in S'} v'_{\gamma'}$ the same argument for the dual (X', X, B', Γ') shows that r is C^∞ on $U' \times X$ with U' as in part (2) of the proposition. Because $(T_B + Q)\mathcal{F}_{B'}v'$ equals $\sigma_{B'}r$, we see that the remaining half of the claim has also been proved. This completes the proof of the proposition. \square

6. THE POISSON FORMULA FOR MILD SINGULARITIES

We apply the considerations of the previous section to the spaces of distributions given below. The notation and operators introduced there will be employed here as well.

Definition 6.1. We will define the following spaces of distributions on X :

$$\mathcal{H}_\infty(X), \mathcal{H}_\infty^+(X), \mathcal{H}_0(X), \mathcal{H}_0^+(X), \mathcal{H}(X), \mathcal{D}_c(X).$$

A distribution u on X belongs to $\mathcal{H}_\infty(X)$ (resp. $\mathcal{H}_0(X)$) if

- (a) u is C^∞ on the region $\|x\| > R$ (resp. $0 < \|x\| < R$) for some $R > 0$, and
- (b) there is some real number p with the property

$$(50) \quad \|x\|^m \partial_v^m u(x) \text{ is } O(\|x\|^p) \text{ for all } v \in X, m \geq 0$$

as $\|x\| \rightarrow \infty$ (resp. as $\|x\| \rightarrow 0$).

A distribution u on X is in $\mathcal{H}(X)$ if

- (a) u is C^∞ on the region $0 \neq x \in X$ and
- (b) it belongs to both $\mathcal{H}_0(X)$ and $\mathcal{H}_\infty(X)$. Note, however, that the p 's that appear for $\|x\| \rightarrow \infty$ and $\|x\| \rightarrow 0$ may be different from each other.

We put $\mathcal{H}_\infty^+(X) = C^\infty(X) \cap \mathcal{H}_\infty(X)$.

A distribution u on X belongs to $\mathcal{H}_0^+(X)$ if it satisfies the three conditions:

- (i) u is compactly supported,
- (ii) the singular support of u is contained in $\{0\}$
- (iii) $u \in \mathcal{H}_0(X)$.

Note that if u belongs to $\mathcal{H}_\infty(X)$, then u is certainly a tempered distribution, and therefore $\mathcal{F}_B u$ is defined.

$\mathcal{D}_c(X)$ is the space of compactly supported distributions on X .

Proposition 6.2. $u \in \mathcal{H}_\infty^+(X) \iff \mathcal{F}_B u \in \mathcal{S}(X') + \mathcal{H}_0^+(X')$.

Proof. Given $x' \in X'$, the function $x \mapsto B(x, x')$, denoted in (29) by $B_{x'}$, will now be denoted simply by x' .

Let $u = f \in \mathcal{H}_\infty^+(X) = C^\infty(X) \cap \mathcal{H}_\infty(X)$, and let $p \in \mathbb{R}$ be as in (50). Choose an integer $k \geq 0$ so that $k - p > \dim X$. The inequality of (50) assumed for $\|x\| \mapsto \infty$ now implies

$$(51) \quad (v')^m \partial_v^{m+r+k} f \in L^1(X) \text{ for all } m \geq 0, r \geq 0, v' \in X', v \in X.$$

From (29), we deduce

$$(52) \quad \partial_{v'}^m v^{m+r+k} \mathcal{F}_B f \in C_0(X') \text{ for all } m \geq 0, r \geq 0, v' \in X', v \in X.$$

From Weyl's commutation relations, one checks that for every $h \in \mathbb{Z}$,

$$(53) \quad \{\partial_{v'}^m v^n : n - m = h, v \in X, v' \in X'\} \text{ and } \{v^n \partial_{v'}^m : n - m = h, v \in X, v' \in X'\}$$

have the same linear span in the Weyl algebra of X' , the ring of differential operators with polynomial coefficients on X' . Putting $h = k, k + 1, \dots$ we deduce that

$$(54) \quad v^{m+r+k} \partial_{v'}^m f(x) \in C_0(X') \text{ for all } m \geq 0, r \geq 0, v' \in X', v \in X$$

We then claim that for all $v' \in X'$ and $m \geq 0$,

- (i) $x' \mapsto \partial_{v'}^m \mathcal{F}_B f(x')$ is continuous at all $0 \neq x' \in X'$, and

- (ii) $|\partial_{v'}^m \mathcal{F}_B f(x')| \leq C(v', m) \|x'\|^{-k-m}$ for all $0 \neq x' \in X'$.
- (iii) $|\partial_{v'}^m \mathcal{F}_B f(x')|$ is $O(\|x'\|^{-h})$ as $x' \rightarrow \infty$ for every $h \in \mathbb{Z}$.

Both (i) and (ii) are deduced from (54) by putting $r = 0$, letting the v 's run through a basis of X , for any fixed choice of $v' \in X'$ and $m \geq 0$. Whereas (iii) is deduced in the same manner by letting r go to infinity in (54). Let $\phi \in C_c^\infty(X')$ so that $\phi^{-1}(1)$ contains a neighbourhood of zero in X' . Then (i) and (ii) imply that $\phi \mathcal{F}_B f$ belongs to $\mathcal{H}_0^+(X')$ whereas (i) and (iii) imply that $(1 - \phi) \mathcal{F}_B f$ belongs to $\mathcal{S}(X')$.

We have now shown \implies of the proposition.

For the reverse implication, it suffices to prove that $\mathcal{F}_{B'} u \in \mathcal{H}_\infty^+(X)$ whenever $u \in \mathcal{H}_0^+(X')$. So let u be such a distribution on X' . Let $p(u)$ be the supremum of the set of $p \in \mathbb{R}$ for which the inequality of (50) is valid for $\|x'\| \rightarrow 0$.

We shall deal first with the case: $p(u) > 0$. Recall that the restriction of u to $X' \setminus \{0\}$ is given by a C^∞ function $f : X \setminus \{0\} \rightarrow \mathbb{C}$. The assumption $p(u) > 0$ implies that f extends as a continuous function $f : X' \rightarrow \mathbb{C}$ such that $f(0) = 0$. In particular, f gives rise to a distribution on X , which we once again denote by f . The distribution $w = u - f$ is supported at 0. By a theorem of L. Schwartz (see [R2], thm.6.25, page 150, and the identity (29) for tempered distributions), the Fourier transform of w is a polynomial on X and therefore belongs to $\mathcal{H}_\infty^+(X)$. So it remains to show that $\mathcal{F}_{B'} f$ is in $\mathcal{H}_\infty^+(X)$. Because f is compactly supported, its Fourier transform is C^∞ , so we only have to verify (50). We see that $v^m f$ is m -times continuously differentiable on all of X' , for all $v \in X$, and so it follows that the distribution $\partial_{v'}^m v^m f$ is a continuous function for all $m \geq 0, v \in X, v' \in X$, and therefore in $L^1(X')$, being compactly supported. The identities (29) now show that $\partial_v^m \mathcal{F}_{B'}(x)$ is $O(\|x\|^{-m})$ as $\|x\| \rightarrow \infty$, as required.

The general case $p(u) + k > 0$ is dealt with by induction on $k \geq 0$. From Weyl's commutation relations, one sees that $p(v.u) \geq 1 + p(u)$ for all $v \in X$. Thus we may assume that $\mathcal{F}_{B'}(v.u)$ is in $\mathcal{H}_\infty^+(X)$ for all $v \in X$. In other words, $\partial_v \mathcal{F}_{B'} u$ belongs to $\mathcal{H}_\infty^+(X)$ for every $v \in X$. From this, it is immediate that $\mathcal{F}_{B'} u$ is itself in $\mathcal{H}_\infty^+(X)$. This completes the proof. \square

Observation 6.3. We intend to express some distributions in the form encountered in Proposition 5.11. For this, it is useful to note that

$$v + f + \mathcal{F}_{B'} v' = v_1 + f_1 + \mathcal{F}_{B'} v'_1 \implies v - v_1 \in C_c^\infty(X) \text{ and } v' - v'_1 \in C_c^\infty(X')$$

where it is assumed that $v, v_1 \in \mathcal{D}_c(X), v', v'_1 \in \mathcal{D}_c(X'), f, f_1 \in \mathcal{S}(X)$. Indeed, by Paley-Wiener-Schwartz, $\mathcal{F}_{B'}(v' - v'_1)$ is C^∞ , and so it follows that $v - v_1$ is C^∞ as well. The same argument after an application of \mathcal{F}_B to both sides shows that $v' - v'_1$ is C^∞ as well.

Proposition 6.4.

- (1) $\mathcal{H}_\infty(X) = \mathcal{D}_c(X) + \mathcal{S}(X) + \mathcal{F}_{B'} \mathcal{H}_0^+(X')$.
- (2) $\mathcal{H}(X) = \mathcal{H}_0^+(X) + \mathcal{S}(X) + \mathcal{F}_{B'} \mathcal{H}_0^+(X')$.
- (3) $\mathcal{H}(X) = \mathcal{H}_\infty(X) \cap \mathcal{F}_{B'} \mathcal{H}_\infty(X')$.
- (4) $\mathcal{F}_B \mathcal{H}(X) = \mathcal{H}(X')$.

Proof. We first note that

- (a) $\mathcal{H}_\infty^+(X) = \mathcal{S}(X) + \mathcal{F}_{B'} \mathcal{H}_0^+(X')$
- (b) $\mathcal{H}_\infty(X) = \mathcal{H}_\infty^+(X) + \mathcal{D}_c(X)$

(c) $\mathcal{H}(X) = \mathcal{H}_0^+(X) + \mathcal{H}_\infty^+(X)$.

Both (b) and (c) are obtained by writing $u = \phi.u + (1 - \phi).u$ for suitable test functions $\phi \in C_c^\infty(X)$, and (a) is just Proposition 6.2. Now part (1) follows from (a) and (b), and part (2) follows from (a) and (c).

We come to part (3). By part (1), we may express $u \in \mathcal{H}_\infty(X)$ as $u = v + f + \mathcal{F}_{B'}v'$ where $v \in \mathcal{D}_c(X)$, $f \in \mathcal{S}(X)$, $v' \in \mathcal{H}_0^+(X')$. Now, if u also belongs to $\mathcal{F}_{B'}\mathcal{H}_\infty(X')$, we see that $u = v_1 + f_1 + \mathcal{F}_{B'}v'_1$ with $v_1 \in \mathcal{H}_0^+(X)$, $f_1 \in \mathcal{S}(X)$, $v'_1 \in \mathcal{D}_c(X')$. By the trivial observation 6.3, we see that $v - v_1 \in C_c^\infty(X)$. Because v_1 belongs to $\mathcal{H}_0^+(X)$, we see that v belongs to the same space as well. It follows that $\mathcal{H}_\infty(X) \cap \mathcal{F}_{B'}\mathcal{H}_\infty(X')$ is the same as $\mathcal{H}_0^+(X) + \mathcal{S}(X) + \mathcal{F}_{B'}\mathcal{H}_0^+(X')$. Part (3) now follows from part (2). Part (4) is immediate from part (3). \square

Lemma 6.5. *Let $f \in \mathcal{H}_\infty^+(X)$. Then $T_B f$ is continuous on $X \times (X' \setminus \Gamma')$. Furthermore, $T_B f(x, x')$ is simply the sum of the t -convergent series $\sum_{\gamma \in \Gamma} f(x + \gamma) \psi_B(\gamma, x')$ for every $x \in X, x' \in X', x' \notin \Gamma'$.*

Proof. The action of X on $\mathcal{S}(X)$ and $C^\infty(X \times X' / \Gamma \times \Gamma')$ given by translations:

$$L_v f(x) = f(x - v) \text{ and } L_v h(x, x') = h(x - v, x') \quad \forall x, v \in X, \forall x' \in X'$$

for all $f \in \mathcal{S}(X)$, $h \in C^\infty(X \times X' / \Gamma \times \Gamma')$ extends in the standard manner to an action on $\mathcal{S}(X)^*$ and $\mathcal{D}(X \times X' / \Gamma \times \Gamma')$ respectively. Furthermore $T_B \circ L_v = L_v \circ T_B$ for all $v \in X$. It follows that T_B commutes with $(1 - L_v)^m$ for all $m \geq 0$.

Now let $f \in \mathcal{H}_\infty^+(X)$. Choose $m \geq 0$ so that $\partial_v^m f$ is $O(\|x\|^{-(1+\dim X)})$. Employing (10), such upper bounds are valid for $g = (1 - L_v)^m f$ as well. Lemma 5.9 shows that the distribution $T_B g$ is a continuous function, and also that $T_B g(x, x')$ is the sum of the absolutely convergent series $\sum_{\gamma \in \Gamma} g(x + \gamma) \psi_B(\gamma, x')$ for all $x \in X, x' \in X'$.

Now take $v \in \Gamma$. Let $w(x, x') = \psi_B(v, x')$ for all $(x, x') \in X \times X'$. By (32), we see that $T_B L_v u = w.T_B u$ for all $u \in \mathcal{S}(X)^*$. It follows that $T_B g = (1 - w)^m T_B f$. The continuity of $T_B f$ on the region $w \neq 1$ follows. That the series in lemma 6.5 is t -convergent has already been remarked in thm. 3.7. The sum of this t -convergent series is $(1 - \psi_B(v, x'))^{-m} T_B g(x, x')$ when $\psi_B(v, x') \neq 1$, by its definition. Now, if $x' \notin \Gamma'$, there is some $v \in \Gamma$ for which $\psi_B(v, x') \neq 1$. This proves the lemma.

That $T_B f$ is C^∞ can be proved by the same method. But it is also a consequence of the theorem below because $\mathcal{H}_\infty^+(X) \subset \mathcal{H}(X)$. \square

Theorem 6.6. *Let $u \in \mathcal{H}(X)$. Let $f : X \setminus \{0\} \rightarrow \mathbb{C}$ (resp. $f' : X' \rightarrow \mathbb{C}$) the C^∞ function obtained by restricting u (resp. $\mathcal{F}_B u$) to the complement of 0 in X (resp in X'). Let $U = X \setminus (\Gamma \setminus \{0\})$ and let $U' = X' \setminus (\Gamma' \setminus \{0\})$. Then*

- (1) $T_B^{reg} u$ is defined
- (2) $T_B^{reg}|_{U \times U'}$ is C^∞ .
- (3) Furthermore $T_B^{reg} u(x, x')$ equals both of the t -convergent sums below
 - (a) $-\psi_B(-x, x') f'(x') + \sum_{0 \neq \gamma \in \Gamma} f(x + \gamma) \psi_B(\gamma, x')$ for all $x \in U, x' \notin \Gamma'$
 - (b) $-f(x) + \sum_{0 \neq \gamma' \in \Gamma'} f'(x' + \gamma') \psi_B(-x, x' + \gamma')$ for all $x \notin \Gamma, x' \in U'$.

Proof. By 6.4(2), we see that u has the form encountered in 5.11. The singular supports of u and $\mathcal{F}_B u$ are contained in $\{0\}$. So parts (1) and (2) follow from 5.11.

The above lemma shows that $T_B^{reg} u$ is given by the sum in (a) when $u \in \mathcal{H}_\infty^+(X)$. If $u \in \mathcal{H}_0^+(X)$, the sum in (a) is finite and thus equals $T_B^{reg} u$ by thm. 5.8. It is clear that $\mathcal{H}_\infty^+(X) + \mathcal{H}_0^+(X) = \mathcal{H}(X)$. Therefore it has been proved that the sum in (a) equals $T_B^{reg} u$ for all $u \in \mathcal{H}(X)$. The remaining equality is obtained by interchanging the roles of u and $\mathcal{F}_B u$. \square

Remark 6.7. With U as given in prop. 5.11 and U' as above, it is clear that the equality of T_B^{reg} with the sum in (a) holds also when $u \in \mathcal{H}_\infty(X)$.

Definition 6.8. We define $\mathcal{L}u = T_B^{reg} u(0, 0)$ for $u \in \mathcal{H}(X)$ and $\mathcal{L}u' = T_{B'}^{reg} u'(0, 0)$ for $u' \in \mathcal{H}(X)$. We note that $\mathcal{L}u = \mathcal{L}\mathcal{F}_B u$ and also that \mathcal{L} is invariant under the action of $\text{Aut}(\Gamma)$.

Observation 6.9. The above theorem contains all the assertions of the theorems in the introduction with one exception: it still has to be shown that $\mathcal{L}u$ is given by the algebraic formula (5) under certain homogeneity assumptions. We address this issue now.

The action of $g \in \text{GL}(X)$ on distributions on $X, X', X \times X'$ is denoted by $\rho(g), \rho'(g), \rho''(g)$ respectively, and $\lambda \mapsto \lambda_X$ denotes the inclusion $\mathbb{R}^\times \hookrightarrow \text{GL}(X)$.

Let $N \in \mathbb{N}$. For a distribution w on $X \times X'$ or X' that is invariant under translation by Γ' , let $U_N w$ be the sum of its translates over all the N -torsion points of X'/Γ' . For $u \in \mathcal{S}(X)^*$ and u_γ as in 5.8 we note that $U_N u_\gamma = N^{\dim X} u_\gamma$ if $\gamma \in N\Gamma$ and zero otherwise. Summing over Γ and simplifying, we see:

$$(a) \quad U_N T_B u = N^{\dim X} \rho''(N_X) T_B \rho(N_X)^{-1} u.$$

The terms u_γ for $\gamma = 0$ (resp $u'_{\gamma'}$ for $\gamma' = 0$ when $u' \in \mathcal{S}(X')^*$) will be denoted by $u(x, 0)$ (resp. $u'(0, x')$). Let $T'_B u = T_B u - u(x, 0)$. We see that (a) can be rewritten as:

$$(b) \quad U_N T'_B u = N^{\dim X} \rho''(N_X) T'_B \rho(N_X)^{-1} u.$$

Now write $U_N w = w + U'_N w$. Subtract $\psi_B(-x, x') \mathcal{F}_B u(0, x')$ from both sides of (b) when $u \in \mathcal{H}(X)$. We obtain:

$$(55) \quad U'_N T'_B u + T_B^{reg} u = N^{\dim X} \rho''(N_X) T_B^{reg} \rho(N_X)^{-1} u \text{ for all } u \in \mathcal{H}(X).$$

All the three terms above are C^∞ at $(0, 0)$ and evaluation at this point gives

$$(56) \quad U'_N T'_B u(0, 0) + \mathcal{L}u = N^{\dim X} \mathcal{L} \rho(N_X)^{-1} u \text{ for all } u \in \mathcal{H}(X).$$

Part 3(a) of Thm. 6.6 has a sum indexed by $0 \neq \gamma \in \Gamma$, and this is precisely $T'_B u(x, x')$. Because $U'_N T'_B u(0, 0)$ is the sum of the $T'_B u(0, x')$ taken over all the nontrivial N -torsion points x' of X'/Γ' we see that (56) is equivalent to (5) under the homogeneity assumption $\rho(\lambda_X u) = \lambda^s u$ for all positive real numbers λ .

Definition 6.10. A distribution u on X is **t-integrable** if there is a pair (ϕ, f) with $\phi \in C_c^\infty(X), f \in \mathcal{S}(X)$ that satisfy

- (1) $\int_X \phi \neq 0$ and $\int_X f \neq 0$,
- (2) $\phi * u \in \mathcal{S}(X)^*$ and $f * (\phi * u) \in L^1(X)$.

$\int_X u$ is then defined as $(\int_X \phi \cdot \int_X f)^{-1} \int_X f * (\phi * u)$ and is seen to be independent of the choice of (ϕ, f) thanks to commutativity.

A function $a : \Gamma \rightarrow \mathbb{C}$ is **t^+ summable** if there is a pair (c, g) of \mathbb{C} -valued functions on Γ , where c has finite support, and g is rapidly decreasing i.e. $|g(n)|$ is $O(\|n\|^{-r})$ for all r , with

- (i) $\Sigma_{\Gamma} c \neq 0$ and $\Sigma_{\Gamma} g \neq 0$
- (ii) $c * a : \Gamma \rightarrow \mathbb{C}$ has polynomial growth, and $g * (c * a)$ belongs to $L^1(\Gamma)$.

$\Sigma_{\Gamma} a$ is then defined in the predictable manner. If a is t -summable, naturally it is clearly t^+ summable. If a is t^+ summable, then the distribution $E(a) = \sum_{\gamma \in \Gamma} a(\gamma) \delta_{\gamma}$ (where δ_{γ} is the Dirac distribution at γ) on X is t -integrable. To see this, choose any $h \in C_c^{\infty}(X)$ with $\int_X h \neq 0$. If the pair (c, g) proves the t^+ summability of a , then the pair (ϕ, f) given by $\phi = h * E(c)$ and $f = h * E(g)$ proves the t -integrability of $E(a)$.

7. GROUPS ACTING ON SETS

The noncommutative situation 7.1. We return to the situation of 2.1 and once again define the canonical extension given the data (a),(b),(c) as stated there, but with one important difference: it is *not* assumed that A is commutative. We will also assume that $\epsilon : A \rightarrow k$ is *surjective*, and denote by \mathfrak{m} its kernel.

Let $\mathcal{W} = \{W : M \subset W \subset N, W \text{ is a } A\text{-submodule, and } \text{Hom}_A(W'/M, k) = 0 \text{ for all } A\text{-submodules } M \subset W' \subset W\}$. We denote by \mathcal{W}_1 the collection of $W \in \mathcal{W}$ for which there is an A -module homomorphism $I_W : W \rightarrow k$ satisfying $I_W|_M = I$.

We first check that if $W_1, W_2 \in \mathcal{W}$, then $W_1 + W_2$ also belongs to \mathcal{W} . Indeed, if $M \subset W' \subset W_1 + W_2$, let $D = W' \cap W_1$. Note that D/M and W'/D are subquotients of W_1/M and W_2/M respectively. It follows that $\text{Hom}_A(D/M, k) = \text{Hom}_A(W'/D, k) = 0$ and this shows that $\text{Hom}_A(W'/M, k) = 0$ as well. Thus $W_1 + W_2 \in \mathcal{W}$. By induction, it follows that \mathcal{W} is closed under finite sums. It is then obvious that \mathcal{W} is closed under arbitrary sums.

We continue by showing that \mathcal{W}_1 has the same property. Let $W_1, W_2 \in \mathcal{W}_1$. By definition, we have A -module homomorphisms $I_1 : W_1 \rightarrow k$ and $I_2 : W_2 \rightarrow k$ that satisfy $I_1|_M = I_2|_M = I$. Let $W' = W_1 \cap W_2$. It follows that $I_1|_{W'} - I_2|_{W'}$ factors through a homomorphism $h : W'/M \rightarrow k$. Now W'/M is a submodule of W_1/M , and so we see that $h = 0$. We deduce that there is a $\tilde{I} : W_1 + W_2 \rightarrow k$ that extends both I_1 and I_2 . It has been shown now that $W_1 + W_2 \in \mathcal{W}_1$. Repeating the same steps as above, we see that \mathcal{W}_1 is closed under arbitrary sums.

We define $M_c : \Sigma\{W : W \in \mathcal{W}_1\}$ and denote by $I_c : M_c \rightarrow k$ the unique A -module homomorphism that restricts to the given $I : M \rightarrow k$.

The observation below is useful for the proposition below, but straightforward from the definitions, and so we skip its proof.

Observation 7.2. Let $W \in \mathcal{W}_1$ and let $I_W : W \rightarrow k$ be the unique A -module homomorphism that extends $I : M \rightarrow k$. Then the canonical extension associated to the data $(A, \epsilon : A \rightarrow k, W \hookrightarrow N, I_W : W \rightarrow k)$ is the same as (M_c, I_c) .

We have seen that $n \in N$ belongs to M_c if and only if $M + An \in \mathcal{W}_1$. We spell out this condition as explicitly as we can below.

For $n \in N$, let $J_n \subset A$ be the left ideal that annihilates its image $\bar{n} \in N/M$. Let $i_n : A/J_n \rightarrow N/M$ be the homomorphism that sends 1 to \bar{n} . The short exact sequence $0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0$ gives $\xi \in \text{Ext}_A^1(N/M, M)$. The functoriality of

Ext in both variables, given $I : M \rightarrow k$ and the above $i_n : A/J_n \rightarrow N/M$ produces an element $\theta_n \in \text{Ext}_A^1(A/J_n, k)$.

We deduce that $n \in \mathcal{W}$ if and only if $\text{Hom}_A(A/J_n, k) = 0$ for all $a \in A$. And $n \in \mathcal{W}_1$ if and only if θ_n vanishes, in addition.

By the long exact sequence of $\text{Ext}_A(\cdot, k)$ one identifies $\text{Hom}_A(A/J_n, k)$ and $\text{Ext}_A^1(A/J_n, k)$ with $\text{Hom}_k(A/\mathfrak{m} + J_n, k)$ and $\text{Hom}_k((\mathfrak{m} \cap J_n/\mathfrak{m}J_n), k)$ respectively. Under this identification, the composite $\mathfrak{m} \cap J_n \rightarrow \mathfrak{m} \cap J_n/\mathfrak{m}J_n \xrightarrow{\theta_n} k$ is easily seen to be $a \mapsto I(an)$ for all $a \in \mathfrak{m} \cap J_n$.

The S in 2.1 is the complement of \mathfrak{m} . The condition $\text{Hom}_A(A/J_n, k) = 0 \iff J_n + \mathfrak{m} = A \iff S \cap J_n \neq \emptyset$. The statement below summarises this discussion.

Observation 7.3. Let $n \in N$. Then $n \in M_c$ if and only if

- (a) $J_{an} + \mathfrak{m} = A$ for all $a \in A$, and
- (b) $I(an) = 0$ for all $a \in \mathfrak{m} \cap J_n$.

When A is commutative, the condition $J_n + \mathfrak{m} = A$ suffices for (a) because J_{an} contains J_n for all $a \in A$. It also suffices for (b) because $\mathfrak{m} \cap J_n = \mathfrak{m}J_n$. We deduce that the (M_c, I_c) given here in 7.1 is consistent with that of 2.1.

We continue with $A = \mathbb{C}[G]$ and $\epsilon : A \rightarrow \mathbb{C}$ the augmentation homomorphism as before. The canonical extension (M_c, I_c) will now be denoted by (M_G, I_G) . If H is a subgroup, we also get the canonical extension when A is replaced by $\mathbb{C}[H]$, and this will naturally be denoted by (M_H, I_H) .

Proposition 7.4. Assume that H is a normal subgroup of G . Then

- (1) M_H is a G -submodule of M_G and $I_G|_{M_H} = I_H$.
- (2) (M_G, I_G) as defined above is also the canonical extension associated to the data $(\mathbb{C}[G], M_H \hookrightarrow N, I_H : M_H \rightarrow \mathbb{C})$.

Proof. It is evident that M_H is a G -module and that I_H is a G -invariant linear functional. By assumption, there are no nonzero H -invariant linear functionals on any H -module W , with $M \subset W \subset M_H$. It follows that there are no nonzero G -invariant linear functionals on any G -module W , with $M \subset W \subset M_H$. It follows that $M_H \in \mathcal{W}_1$. The rest of the proposition follows from 7.2. □

From now on, we concentrate on the case: X is a set equipped with G -action, $M = L^1(X) \subset N = \mathbb{C}^X$ and $I : M \rightarrow \mathbb{C}$ is Σ_X . When $f : X \rightarrow \mathbb{C}$ belongs to M_G , we will say the series $\Sigma_X f$ is G -summable.

Very specifically, we take $X = \mathbb{Q}^k$ equipped with the action of $G = \mathbb{Q}^k \rtimes \text{GL}_k(\mathbb{Q})$. The center of $\text{GL}_k(\mathbb{Q})$ is denoted simply by $\mathbb{Q}^\times \subset \text{GL}_k(\mathbb{Q})$. We put $H_1 = \mathbb{Q}^k$ and $H_2 = \mathbb{Q}^\times$. We put $H = \mathbb{Q}^k \rtimes \mathbb{Q}^\times$. In view of the normality of \mathbb{Q}^k and H in G , the preceding proposition shows we have the chain of spaces

$$L^1(\mathbb{Q}^k) \subset L^1(\mathbb{Q}^k)_{\mathbb{Q}^k} \subset L^1(\mathbb{Q}^k)_H \subset L^1(\mathbb{Q}^k)_G.$$

$L^1(\mathbb{Q}^k)_{\mathbb{Q}^k}$ is the space of t -summable series, by very definition.

The precise relation between H -summability as above and h -summability in the section 4 has not been worked out. The following fact, simple to verify, is left to the reader.

Observation 7.5. Let $f : \mathbb{R}^k \setminus \{0\} \rightarrow \mathbb{C}$ be a C^∞ function homogeneous of type $(-s, \epsilon)$. Assume $f \neq 0$. Then f is H -summable if and only if $(-s, \epsilon) \notin \{(-k + m, (-1)^m) : m \geq 0, m \in \mathbb{Z}\}$.

Example 7.6. We take $m = 1$. For $k = 1$, this says that $h : \mathbb{Q} \rightarrow \mathbb{C}$ given by $h(x) = x/|x|$ if $0 \neq x \in \mathbb{Z}$, and $f(x) = 0$ for all other rational numbers x is not H -summable. This may also be seen very directly in the following manner. Assume the contrary. Then we see that

$$\sum_{\mathbb{Q}} h = \sum_{x \in \mathbb{Q}} h(x) = \sum_{x \in \mathbb{Q}} h(ax + b)$$

for all $a \in \mathbb{Q}^\times, b \in \mathbb{Q}$. Taking $a = -1, b = 0$ we see that $\sum_{\mathbb{Q}} h = 0$. Taking $a = 1, b = 1$ we see that the sum of the series $x \mapsto h(x + 1) - h(x)$, indexed by $x \in \mathbb{Q}$ is zero, which contradicts the fact that this sum is 2!

For $m = 1, k = 2$, the function $f(x, y) = x/(x^2 + y^2)$ has already occurred in the counterexample given in the proof of 4.7(2).

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REFERENCES

- [1] Apostol, T.M. “Mathematical analysis” Addison-Wesley series in mathematics, 1974.
- [2] Apostol, T.M. “On the Lerch Zeta function” Pacific J. Math. Vol.1 No.2 (1951), 161-167.
- [3] Cassels, J. and Frohlich, A. “Algebraic Number Theory”, Academic Press, 1967.
- [4] Gerardin, P. and Li, W. “Twisted Dirichlet Series and Distributions” Composition Math., tome 73, no.3(1990), p.271-293.
- [5] Hormander, L. “The analysis of linear partial differential operators” Grundlehren 256, Springer-Verlag, 1990.
- [6] Hardy, G.H. “Divergent Series” Oxford, Clarendon Press, 1949.
- [7] Lang, S. “Algebraic Number Theory” GTM 110, Springer-Verlag 1994.
- [8] Lagarias, J. and Li, W. “The Lerch Zeta function I. Zeta Integrals.” Forum Mathematicum, vol.24, 2012, 1-48.
- [9] Lion, G. and Vergne, M. “The Weil representation, Maslov index, and theta series” Progress in mathematics vol. 6, Birkhauser, 1980.
- [10] Mumford, D. “Tata lectures on theta” I,III. Progress in mathematics ; Birkhauser, 1983,1991.
- [11] Miller, S. and Schmid, W. “Distributions and analytic continuation of Dirichlet series”. J. Funct. Anal. 214 (2004), no. 1, 155-220.
- [12] Sondow, J. “Analytic continuation of Riemann’s zeta function and values at negative integers via Euler’s transformation of series.” Proc. Amer. Math. Soc. 120 (1994), no. 2, 421-424.
- [13] Rudin, W. “Principles of mathematical analysis” McGraw-Hill, 1976.
- [14] Rudin, W. “Functional Analysis” Tata McGraw Hill 1974.
- [15] Weil, A. “Basic Number Theory” Grundlehren Bd 144, Springer-Verlag 1974.
- [16] Weil, A. “Sur certains groupes d’opérateurs unitaires.” Acta Math. 111(1964)p143-211.

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